PHYS 110B June 11 Discussion
Spring 2004, Solutions by David Pace
Any referenced equations are from Griffiths
Problem statements are paraphrased

[1.] Problem 11.22 from Griffiths

A magnetic dipole radio antenna is h meters tall. The actual dipole part of the antenna has radius b and its axis is positioned vertically (i.e. the magnetic dipole moment is directed vertically). The station using this antenna broadcasts at angular frequency ω at an averaged power P.

(a) What is the intensity of the radiation from this antenna on the ground a distance R from its base? Assume that $b \ll c/\omega \ll h$.

(b) An engineer took a measurement of the intensity on the ground at the base of the antenna and found this to be a very low value. Where (again, on the ground) should the engineer have taken the measurement such that the maximum intensity would have been observed? What is this maximum intensity?

(c) The station outputs 35 kW at a frequency of 90 MHz and from the problem statement $b = 6$ cm and $h = 200$ m. If the city limits radio emissions to a maximum of 200 μW/cm^2, then is this station in compliance?

Solution

(a) Begin by understanding what the allowed assumption means. Radio waves in air are essentially the same as they would be in vacuum. This means that $\omega/k = c$, with c being the speed of light. Also recall that $k = 2\pi/\lambda$.

$$\frac{c}{\omega} = \frac{1}{k} = \frac{\lambda}{2\pi} \tag{1}$$

So the assumption is,

$$b \ll \frac{\lambda}{2\pi} \ll h \tag{2}$$

which is the usual wavelength restriction. This simplifies the problem by allowing us to neglect issues that arise when the system is as large as the wavelength of the radiation it emits. This amounts to the three approximations used in Griffiths section 11.1.2.

The intensity of radiation from a magnetic dipole is given by,

$$\langle \vec{S} \rangle = \left(\frac{\mu_o m_o^2 \omega^4}{32\pi^2 c^3} \right) \frac{\sin^2\theta}{r^2} \, \hat{r} \qquad 11.39 \tag{3}$$

where m_o is the magnitude of the magnetic dipole moment. This relates to the current in the dipole, which is not known for this problem, so we leave it as m_o for now.

The vector $\vec{r}$ points from the dipole antenna (origin) to the point of observation on the ground. A drawing on your page is probably opposite this, but since the dipole itself does not know "up" or "down", it does not matter. Take the magnitude of (3) and then solve for r through geometry.

$$\sin\theta = \frac{R}{r} \tag{4}$$

$$r = \sqrt{h^2 + R^2} \tag{5}$$

The intensity of radiation at a distance R from the base of the antenna is,

$$|\langle \vec{S} \rangle| = \left(\frac{\mu_o m_o^2 \omega^4}{32\pi^2 c^3}\right)\left(\frac{R^2}{h^2+R^2}\right)\left(\frac{1}{h^2+R^2}\right) \tag{6}$$

$$= \left(\frac{\mu_o m_o^2 \omega^4}{32\pi^2 c^3}\right)\frac{R^2}{(h^2+R^2)^2} \tag{7}$$

We still have the m_o to deal with. The problem tells us the total radiated power is P. The general expression for total radiated power from a magnetic dipole is,

$$\langle P \rangle = \frac{\mu_o m_o^2 \omega^4}{12\pi c^3} \qquad \text{Eq. 11.40} \tag{8}$$

Now it is possible to put the solution in terms of parameters given in the problem statement,

$$|\langle \vec{S} \rangle| = S = \left(\frac{3P}{8\pi}\right)\frac{R^2}{(h^2+R^2)^2} \tag{9}$$

(b) Notice from (3) that at a location directly beneath the antenna we have $\theta = 0$ and therefore should measure an intensity of zero. At the very least, taking a measurement at this location will provide the lowest possible measure of intensity.

To ensure safety, the radiation measurement should be taken at the point on the ground where the maximum intensity will be observed. This is found by,

$$\frac{dS}{dR} = 0 = \left(\frac{3P}{8\pi}\right)\frac{d}{dR}\frac{R^2}{(h^2+R^2)^2} \tag{10}$$

$$= \frac{(h^2+R^2)^2 2R - R^2(2)(h^2+R^2)2R}{(h^2+R^2)^4} \tag{11}$$

$$= 2R(h^2+R^2) - 4R^3 \tag{12}$$

$$h^2 = R^2 \tag{13}$$

$$R = h \tag{14}$$

At a distance $R = h$ away from the tower the maximum intensity will be measured on the ground. Referring back to (9) this intensity is,

$$S = \frac{3P}{8\pi}\frac{h^2}{(h^2+h^2)^2} \tag{15}$$

$$= \frac{3P}{32\pi h^2} \tag{16}$$

(c) The object of this problem is to insert the given parameters into (16) and determine whether the maximum intensity output to the ground by the radio antenna is above the limit. The frequency does not matter because we absorbed that into the total power radiated.

$$S = \frac{3(35 \times 10^3)}{32\pi(200)^2} \approx 0.026 \frac{W}{m^2} \tag{17}$$

This is an emission of approximately $2.6\ \mu$W/cm^2, which is well below the city's imposed limit.

[2.] Non-Griffiths Problem

Two point particles of equal charge q are fixed at opposite ends of a rigid rod (length $2l$). This system rotates about an axis that is perpendicular to the rod and passes through its center. The angular velocity of this rotation is $\omega = \zeta/2$.

(a) Find the electric and magnetic dipole moments of this system.

(b) Based on the models for radiation presented in this class, what is the power radiated from this system? HINT: The previous question pertains directly to this one.

(c) In what way can you expand upon the radiative models from class to change the solution to the previous question?

Solution

(a) The expression for the electric dipole moment in a system of n charges is,

$$\vec{p} = \sum_i^n q_i \vec{d_i} \tag{18}$$

This is a vector sum and since the charges are identical the q may be factored out.

$$\vec{p} = q(\vec{r}_1 + \vec{r}_2) \tag{19}$$

The next step is to recognize that these particles are at opposite ends of the rod. Imagine that the rod lies along the x-axis with its center on the origin. In this case,

$$\vec{p} = q(l\hat{x} - l\hat{x}) \tag{20}$$

$$= 0 \tag{21}$$

NOTE: We know from chapter three of Griffiths (page 152 in particular) that the dipole moment of a collection of charges with a non-zero total charge depends upon the choice of origin. At this time it is useful to keep the origin at the center of the rod where the problem is most greatly simplified. This will lead to (hopefully) interesting points in the next parts of this problem.

Even as the rod spins, the vectors indicating the positions of the charges will always cancel. The electric dipole moment of this system is zero.

The magnetic dipole moment is given by $m\vec{m} = I\vec{a}$, where I is the current in the loop formed by the motion of the particles and $\vec{a}$ is the vector area of this loop. In this discrete case we can write out the current as,

$$I = \frac{\Delta q}{\Delta t} \tag{22}$$

$$= \frac{2q}{T} \tag{23}$$

where T is the period of this system. It takes this amount of time for both charges to pass through a specific point on their path.
The period is known because the angular frequency is given.

$$T = \frac{1}{f} \qquad \omega = 2\pi f \tag{24}$$

$$= \frac{2\pi}{\omega} \tag{25}$$

$$= \frac{4\pi}{\zeta} \tag{26}$$

The vector area of the current loop that results from the rotation of the rod is,

$$\vec{a} = \pi l^2 \, \hat{z} \tag{27}$$

where the direction is chosen, just as putting the rod's center on the origin was chosen.
The magnetic dipole moment is,

$$\vec{m} = 2q\frac{\zeta}{4\pi}\left(\pi l^2\right) \, \hat{z} \tag{28}$$

$$= \frac{q\zeta l^2}{2} \, \hat{z} \tag{29}$$

(b) Working with dipole radiation we have come to the following results for the power radiated by electric and magnetic oscillating systems, respectively,

$$P_e \cong \frac{\mu_o \ddot{p}^2}{6\pi c} \qquad \text{Eq. 11.60} \tag{30}$$

$$P_m \cong \frac{\mu_o \ddot{m}^2}{6\pi c^3} \tag{31}$$

The magnetic dipole expression is given at the end of problem 11.12.
In the case of dipoles the radiated power depends on the second time derivative of the dipole moment. These moments are both constant in time for this problem. Based on our dipole model study of radiation we conclude that there is no radiation coming from this system.

(c) There is good reason for the $\cong$ sign in the expressions directly above. The scalar and vector potentials of a charge distribution can be written in terms of the moments. This results in the monopole, dipole, and quadrupole moments, in addition to the infinite higher order moments. Just as in the case of the potential these higher order moments tend to make smaller and smaller contributions to the radiation. If the dipole terms are constant (or if for another reason their second time derivatives are zero), then there is no dipole radiation and the higher order terms may be significant.

The electric quadrupole term in this system does contribute to the radiation. If we were to consider this term, then we would have a non-zero solution to part (b).

[3.] Problem 11.23 from Griffiths

The magnetic north pole of the earth is not at the same location as the geographic north pole. An angle of $11°$ separates these poles. Since the earth rotates about an axis containing the geographic north pole the vector representing the magnetic pole is changing in time. The earth must then be emitting magnetic dipole radiation.

(a) Find an expression for the total power radiated by the earth. Put this in terms of Ψ (the angle between the geographic and magnetic poles), M (the magnitude of the earth's magnetic dipole moment), and ω (the angular velocity of the earth's rotation).

(b) Make an estimate of the magnetic dipole moment, M, of the earth using the fact that the magnetic field at the equator is approximately 0.5 Gauss.

(c) Determine a numerical estimate for the power radiated by the earth.

(d) Consider a rotating neutron star (pulsar). This star has radius $d \approx 10$ km, rotational period $T \approx 10^{-3}$ s, and a surface magnetic field of $B_o \approx 10^8$ T. Estimate the power radiated from a such a star.

Solution

(a) The power radiated by a magnetic dipole is given in (31). It is necessary to solve for the magnetic dipole moment of the earth. The z-component of this moment is $M_z = M\cos\Psi$. The magnitude of the xy-component is $M_{xy} = M\sin\Psi$. Due to the rotation in the xy plane the moment is written completely as,

$$\vec{M}_{earth} = M\sin\Psi[\cos(\omega t)\ \hat{x} + \sin(\omega t)\ \hat{y}] + M\cos\Psi\ \hat{z} \tag{32}$$

$$= M\left[\sin\Psi(\cos(\omega t)\ \hat{x} + \sin(\omega t)\ \hat{y}) + \cos\Psi\ \hat{z}\right] \tag{33}$$

The second order time derivative of (33) is needed for the radiated power expression. Do not be confused by the Ψ dependent terms; these are constant in time.

$$\dot{\vec{M}}_{earth} = M\sin\Psi(-\omega\sin(\omega t)\ \hat{x} + \omega\cos(\omega t)\ \hat{y}) \tag{34}$$

$$\ddot{\vec{M}}_{earth} = M\sin\Psi(-\omega^2\cos(\omega t)\ \hat{x} - \omega^2\sin(\omega t)\ \hat{y}) \tag{35}$$

Squaring the result of (35) gives,

$$\left(\ddot{\vec{M}}_{earth}\right)^2 = M^2 \sin^2\Psi(\omega^4\cos^2(\omega t) + \omega^4 \sin^2(\omega t)) \tag{36}$$

$$= M^2\omega^4 \sin^2\Psi \tag{37}$$

since $(\vec{k})^2 = \vec{k}\cdot\vec{k}$, where $\vec{k}$ represents any generic vector.
The total power radiated is,

$$P = \frac{\mu_o M^2 \omega^4 \sin^2\Psi}{6\pi c^3} \tag{38}$$

(b) Begin with the field of a magnetic dipole.

$$\vec{B}_{dip} = \frac{\mu_o M}{4\pi r^3}[2\cos\theta\ \hat{r} + \sin\theta\ \hat{\theta}] \tag{39}$$

The problem tells us the value of the magnetic field at the equator. This location is where $\theta = \pi/2$. Rewriting the field at this angle gives,

$$\vec{B}_{dip} = \frac{\mu_o M}{4\pi r^3}\ \hat{\theta} \tag{40}$$

Put this in terms of magnitude and solve for the magnetic dipole moment,

$$M_{earth} = \frac{4\pi r^3 B}{\mu_o} \tag{41}$$

$$= \frac{4\pi(6.4\times10^6)^3(5\times10^{-5})}{4\pi\times10^{-7}} \tag{42}$$

$$\approx 1.3\times10^{23}\ \text{Am}^2 \tag{43}$$

where r is the radius of the earth.

(c) The solution is found by plugging numerical values into (31). Having solved for the magnitude of the dipole moment, the only remaining value to calculate is the angular frequency of the earth's rotation. The earth makes on complete revolution every day (approximately), so the frequency is,

$$f = \frac{1\ \text{revolution}}{1\ \text{day}\times 24\frac{\text{hours}}{\text{day}}\times 60\frac{\text{minutes}}{\text{hour}}\times 60\frac{s}{\text{minute}}} \tag{44}$$

$$\approx 1.2\times10^{-5}\ Hz \tag{45}$$

$$\omega = 2\pi f \approx 7.3\times10^{-5} \tag{46}$$

Using all the proper numerical values in the expression for power radiated provides,

$$P = \frac{4\pi\times10^{-7}(1.3\times10^{23})^2(7.3\times10^{-5})^4\sin^2(11^\circ)}{6\pi(3\times10^8)} \tag{47}$$

$$\approx 4\times10^{-5}\ W \tag{48}$$

and this is the value that Griffiths gives in the text.

(d) Use (41) to find the value of the magnetic dipole moment of the neutron star.

$$M = \frac{4\pi(10 \times 10^3)^3 10^8}{4\pi \times 10^{-7}} \tag{49}$$

$$= 10^{27} \text{ Am}^2 \tag{50}$$

In this case the magnetic field is so strong that it doesn't matter whether the given surface value is measured at the equator.

The angular frequency is found using the given period $T = 10^{-3}$ s,

$$T = \frac{1}{f} = 10^{-3} \tag{51}$$

$$f = 10^3 \tag{52}$$

$$\omega = 2\pi \times 10^3 \tag{53}$$

Putting all of this information into the radiated power expression,

$$P = \frac{4\pi \times 10^{-7}(10^{27})^2(2\pi \times 10^3)^4}{6\pi(3 \times 10^8)^3} \tag{54}$$

$$\approx 3.8 \times 10^{36}\ W \tag{55}$$

The solution that Griffiths gives in the text is 2×10^{36} W. The reason for this is that I neglected the $\sin^2 \Psi$ term in the calculation above. If one considers this squared sine term, then it is reasonable to say that it averages to a factor of $1/2$ (i.e. $\langle \sin^2 \theta \rangle = 1/2$). This works because we are looking only for an approximate solution and without knowing exactly what Ψ is for the neutron star the factor of $1/2$ introduces acceptable error into our solution.