

PHYS 110B - HW #4

Spring 2004, Solutions by David Pace

Any referenced equations are from Griffiths

Problem statements are paraphrased

[1.] Problem 8.2 from Griffiths

Reference problem 7.31 (figure 7.43).

(a) Let the charge on the ends of the wire be zero at $t = 0$. Find the electric and magnetic fields in the gap, $\vec{E}(s, t)$ and $\vec{B}(s, t)$.

(b) Find the energy density and Poynting vector in the gap. Verify equation 8.14.

(c) Solve for the total energy in the gap; it will be time-dependent. Find the total power flowing into the gap by integrating the Poynting vector over the relevant surface. Verify that the input power is equivalent to the rate of increasing energy in the gap. (Griffiths Hint: This verification amounts to proving the validity of equation 8.9 in the case where $W = 0$.)

Solution

(a) The electric field between the plates of a parallel plate capacitor is known to be (see example 2.5 in Griffiths),

$$\vec{E} = \frac{\sigma}{\epsilon_0} \hat{z} \quad (1)$$

where I define \hat{z} as the direction in which the current is flowing.

We may assume that the charge is always spread uniformly over the surfaces of the wire. The resultant charge density on each "plate" is then time-dependent because the flowing current causes charge to pile up. At time zero there is no charge on the plates, so we know that the charge density increases linearly as time progresses.

$$\sigma(t) = \frac{It}{\pi a^2} \quad (2)$$

where a is the radius of the wire and It is the total charge on the plates at any instant.

The electric field between the ends of the wire is,

$$\vec{E} = \frac{It}{\epsilon_0 \pi a^2} \hat{z} \quad (3)$$

The magnetic field is found from Ampere's law,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} \quad (4)$$

where the enclosed current may be due to the displacement term; as it is in this problem.

The displacement current through the gap must be in the \hat{z} direction (the only proper current flow is through the wire). This, coupled with using the displacement current leads to,

$$B_\phi(2\pi s) = \mu_0 I_{d,enc} \quad (5)$$

Solving for the displacement current gives,

$$I_{d,enc} = \int \vec{J}_d \cdot d\vec{a} \quad (6)$$

$$= \int \left(\epsilon_o \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{a} \quad (7)$$

$$= \int_0^s \int_0^{2\pi} \left(\epsilon_o \frac{I}{\epsilon_o \pi a^2} \hat{z} \right) \cdot s \, ds \, d\phi \, \hat{z} \quad (8)$$

$$= \frac{I s^2}{a^2} \quad (9)$$

Now the magnetic field is found from (5),

$$\vec{B} = \frac{\mu_o I s}{2\pi a^2} \hat{\phi} \quad (10)$$

(b) The energy density is given in terms of the fields in the gap,

$$u_{em} = \frac{1}{2} \left(\epsilon_o E^2 + \frac{B^2}{\mu_o} \right) \quad \text{Eq. 8.13} \quad (11)$$

$$= \frac{1}{2} \left(\epsilon_o \frac{I^2 t^2}{\epsilon_o^2 \pi^2 a^4} + \frac{1}{\mu_o} \frac{\mu_o^2 I^2 s^2}{4\pi^2 a^4} \right) \quad (12)$$

$$= \frac{1}{2} \left(\frac{I^2}{\pi^2 a^4} \right) \left[\frac{t^2}{\epsilon_o} + \frac{\mu_o s^2}{4} \right] \quad (13)$$

noting that $E^2 = \vec{E} \cdot \vec{E}$.

The Poynting vector in the gap is,

$$\vec{S} = \frac{1}{\mu_o} \vec{E} \times \vec{B} \quad \text{Eq. 8.10} \quad (14)$$

$$= \frac{1}{\mu_o} \left(\frac{I t}{\epsilon_o \pi a^2} \hat{z} \times \frac{\mu_o I s}{2\pi a^2} \hat{\phi} \right) \quad (15)$$

$$= -\frac{I^2 t s}{2\epsilon_o \pi^2 a^4} \hat{s} \quad (16)$$

The Poynting vector represents energy flow. Taking special note of the direction found above we see that the energy is flowing into the gap.

Verification of equation 8.14 follows ($u_{mech} = 0$),

$$\frac{\partial}{\partial t} (u_{mech} + u_{em}) = -\vec{\nabla} \cdot \vec{S} \quad (17)$$

$$\frac{\partial}{\partial t} \left(\frac{I^2}{2\pi^2 a^4} \right) \left[\frac{t^2}{\epsilon_o} + \frac{\mu_o s^2}{4} \right] = -\frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{-I^2 t s}{2\epsilon_o \pi^2 a^4} \right) \quad (18)$$

$$\frac{I^2 t}{\epsilon_o \pi^2 a^4} = -\frac{1}{s} \left(-\frac{I^2 t s}{\epsilon_o \pi^2 a^4} \right) \quad (19)$$

$$= \frac{I^2 t}{\epsilon_o \pi^2 a^4} \quad (20)$$

(c) Since we have the energy density in the gap we are ready to determine the total energy.

$$U_{em} = \frac{1}{2} \int \left(\epsilon_o E^2 + \frac{B^2}{\mu_o} \right) d\tau \quad \text{Eq. 8.5} \quad (21)$$

$$= \frac{1}{2} \int_0^a \int_0^{2\pi} \int_0^w \left(\frac{I^2}{\pi^2 a^4} \right) \left[\frac{t^2}{\epsilon_o} + \frac{\mu_o s^2}{4} \right] s ds d\phi dz \quad (22)$$

$$= \frac{I^2}{2\pi^2 a^4} (2\pi w) \int_0^a \left[\frac{t^2 s}{\epsilon_o} + \frac{\mu_o s^3}{4} \right] ds \quad (23)$$

$$= \frac{I^2 w}{\pi a^4} \left[\frac{t^2 a^2}{2\epsilon_o} + \frac{\mu_o a^4}{16} \right] \quad (24)$$

$$= \frac{I^2 w}{2\pi a^2} \left[\frac{t^2}{\epsilon_o} + \frac{\mu_o a^2}{8} \right] \quad (25)$$

The problem tells us to determine the total power flowing into the gap by integrating the Poynting vector over the surface enclosing it. This is the cylindrical surface occurring at $s = a$. Technically, this also includes the circular surfaces at each of the plates, but for these surfaces the product $\vec{S} \cdot d\vec{a} = 0$ so they do not contribute to the solution.

$$\text{Power in} = \int_0^{2\pi} \int_0^w \vec{S} \cdot s d\phi dz \hat{s} \quad (26)$$

$$= \int_0^{2\pi} \int_0^w -\frac{I^2 t s}{2\epsilon_o \pi^2 a^4} s d\phi dz \quad (27)$$

$$= -\frac{I^2 t a^2}{2\epsilon_o \pi^2 a^4} (2w\pi) \quad (28)$$

$$= -\frac{I^2 t w}{\epsilon_o \pi a^2} \quad (29)$$

Finally, we essentially want to verify equation 8.9,

$$\frac{dW}{dt} = -\frac{d}{dt} \int_{\mathcal{V}} \frac{1}{2} \left(\epsilon_o E^2 + \frac{1}{\mu_o} B^2 \right) d\tau - \frac{1}{\mu_o} \oint_{\mathcal{S}} \left(\vec{E} \times \vec{B} \right) \cdot d\vec{a} \quad (30)$$

where this represents the rate at which work is done on a collection of charges in the volume \mathcal{V} , that is enclosed by the surface \mathcal{S} .

Since there are no charges in the gap, $W = 0$ and $dW/dt = 0$. The equation to verify is,

$$\frac{d}{dt} \int_{\mathcal{V}} \frac{1}{2} \left(\epsilon_o E^2 + \frac{1}{\mu_o} B^2 \right) d\tau = -\frac{1}{\mu_o} \oint_S \left(\vec{E} \times \vec{B} \right) \cdot d\vec{a} \quad (31)$$

$$\frac{d}{dt} U_{em} = -\oint_S \vec{S} \cdot d\vec{a} \quad (32)$$

we have solved for U_{em} in (25), and the integral on the right hand side of the above equation is the power flowing into the gap, given by (29). Continuing with the verification of equation 8.9,

$$\frac{d}{dt} \frac{I^2 w}{2\pi a^2} \left[\frac{t^2}{\epsilon_o} + \frac{\mu_o a^2}{8} \right] = - \left(-\frac{I^2 t w}{\epsilon_o \pi a^2} \right) \quad (33)$$

$$\frac{I^2 t w}{\epsilon_o \pi a^2} = \frac{I^2 t w}{\epsilon_o \pi a^2} \quad (34)$$

this equation is verified.

[2.] Problem 8.4 from Griffiths

(a) Consider two point charges, of equal charge q , that are separated by a distance $2a$. Integrate the Maxwell stress tensor for this system over the plane separating these charges in order to determine the force of one on the other.

(b) Repeat (a) for charges of opposite sign.

Solution

(a) Generally, the force on charges in some volume \mathcal{V} is,

$$\vec{F} = \oint_S \vec{T} \cdot d\vec{a} - \epsilon_o \mu_o \frac{d}{dt} \int_{\mathcal{V}} \vec{S} d\tau \quad \text{Eq. 8.22} \quad (35)$$

In this problem there is no time dependence. The term involving the time derivative of the Poynting vector is zero.

There are no magnetic fields in this problem so the Maxwell stress tensor is,

$$\vec{T} = \epsilon_o \begin{pmatrix} E_x^2 - \frac{E^2}{2} & E_x E_y & E_x E_z \\ E_y E_x & E_y^2 - \frac{E^2}{2} & E_y E_z \\ E_z E_x & E_z E_y & E_z^2 - \frac{E^2}{2} \end{pmatrix} \quad (36)$$

The expression in (36) was given directly in lecture on 5-5-2004, but it may also be derived from the general form of the stress tensor.

$$T_{ij} \equiv \epsilon_o \left(E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right) + \frac{1}{\mu_o} \left(B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right) \quad \text{Eq. 8.19} \quad (37)$$

Label the charges 1 and 2. I will draw the enclosed region around charge 2 and determine the force that 1 exerts on it. Setting the origin of this system on charge 1, the electric field due to this charge is given by,

$$\vec{E}_1 = \frac{q}{4\pi\epsilon_0 r^2} \hat{r} \quad (38)$$

Writing this in terms of Cartesian coordinates, and setting $x = -a$ since we are only interested in the electric field along the plane between the charges gives,

$$\vec{E}_1 = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{(-a)^2 + y^2 + z^2} \right) \left(\frac{-a\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{(-a)^2 + y^2 + z^2}} \right) \quad (39)$$

$$= \frac{q}{4\pi\epsilon_0} \left(\frac{1}{a^2 + y^2 + z^2} \right) \left(\frac{-a\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{a^2 + y^2 + z^2}} \right) \quad (40)$$

using $r = \sqrt{a^2 + y^2 + z^2}$ as the distance from charge 1 to the plane. The $-a\hat{x}$ term comes from my setting the charge at the origin and placing the other charge at $x = -2a$.

The stress tensor depends on the total field in the system. As such, the effects of the second charge must be included. The symmetry in this problem allows the electric field due to the second charge to be written immediately after a translation in the x coordinate. The electric field of charge 2 is the same as that due to charge 1, except that $x \rightarrow x + 2a$.

$$\vec{E}_2 = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{(-a + 2a)^2 + y^2 + z^2} \right) \left(\frac{(-a + 2a)\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{(-a + 2a)^2 + y^2 + z^2}} \right) \quad (41)$$

$$= \frac{q}{4\pi\epsilon_0} \left(\frac{1}{a^2 + y^2 + z^2} \right) \left(\frac{a\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{a^2 + y^2 + z^2}} \right) \quad (42)$$

Add these to get the total electric field along the plane between them.

$$\vec{E}_{tot} = \vec{E}_1 + \vec{E}_2 \quad (43)$$

$$= \frac{q}{4\pi\epsilon_0} \left(\frac{1}{(a^2 + y^2 + z^2)^{3/2}} \right) [(-a + a)\hat{x} + (y + y)\hat{y} + (z + z)\hat{z}] \quad (44)$$

$$= \frac{q}{2\pi\epsilon_0} \left(\frac{1}{(a^2 + y^2 + z^2)^{3/2}} \right) [y\hat{y} + z\hat{z}] \quad (45)$$

This provides all the information needed to completely write out the stress tensor. First, note the following,

$$E_x = 0 \quad (46)$$

$$E_y = \frac{q}{2\pi\epsilon_0} \left(\frac{y}{(a^2 + y^2 + z^2)^{3/2}} \right) \quad (47)$$

$$E_z = \frac{q}{2\pi\epsilon_0} \left(\frac{z}{(a^2 + y^2 + z^2)^{3/2}} \right) \quad (48)$$

$$E^2 = \frac{q^2}{4\pi^2\epsilon_0^2} \left(\frac{y^2 + z^2}{(a^2 + y^2 + z^2)^3} \right) \quad (49)$$

Referring back to (35), the force on charge 2 due to 1 may be written,

$$\vec{F} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \vec{T} \cdot dy dz \hat{x} \quad (50)$$

where the \hat{x} represents the direction of the surface normal vector from the surface enclosing charge 2.

In the following steps I have included the fact that $E_x = 0$ in order to simplify.

$$\vec{T} \cdot \hat{x} = \epsilon_o \begin{pmatrix} -\frac{E^2}{2} & 0 & 0 \\ 0 & E_y^2 - \frac{E^2}{2} & E_y E_z \\ 0 & E_z E_y & E_z^2 - \frac{E^2}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (51)$$

$$= \epsilon_o \left[-\frac{E^2}{2} \hat{x} + 0\hat{y} + 0\hat{z} \right] \quad (52)$$

$$= -\frac{\epsilon_o}{2} \cdot \frac{q^2}{4\pi^2 \epsilon_o^2} \left(\frac{y^2 + z^2}{(a^2 + y^2 + z^2)^3} \right) \hat{x} \quad (53)$$

$$= -\frac{q^2}{8\pi^2 \epsilon_o} \left(\frac{y^2 + z^2}{(a^2 + y^2 + z^2)^3} \right) \hat{x} \quad (54)$$

Putting this back into the integral expression for the force provides,

$$\vec{F} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} -\frac{q^2}{8\pi^2 \epsilon_o} \left(\frac{y^2 + z^2}{(a^2 + y^2 + z^2)^3} \right) dy dz \hat{x} \quad (55)$$

and it is seen that the final answer will put the force in either the positive or negative \hat{x} . If this were not the case, then we would already know our solution to be incorrect.

The integral in (55) is non-trivial. Since there is no x dependence in the expression for the force along the plane we can use change of coordinate system to obtain a better integral with which to work. Consider the yz plane to be in cylindrical coordinates. If we let $s = \sqrt{y^2 + z^2}$ and then incorporate the ϕ direction into the integral we get,

$$\vec{F} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} -\frac{q^2}{8\pi^2 \epsilon_o} \left(\frac{y^2 + z^2}{(a^2 + y^2 + z^2)^3} \right) dy dz \hat{x} \quad (56)$$

$$= \int_0^{\infty} \int_0^{2\pi} -\frac{q^2}{8\pi^2 \epsilon_o} \left(\frac{s^2}{(a^2 + s^2)^3} \right) s ds d\phi \hat{x} \quad (57)$$

where the new integral is the result of starting over with the geometry considerations and not a mathematical change of variable. Notice that there is an extra factor of s that comes from the da term in cylindrical coordinates. The \hat{x} dependence is left alone because it came from the tensor work and the change to cylindrical coordinates was made after solving this part.

In this case the integral over s is solved using a change of variable and then an integral table.

$$\int_0^\infty \frac{s^3}{(a^2 + s^2)^3} ds \rightarrow \text{Let } u = s^2 \quad du = 2s \, ds \quad (58)$$

$$= \int_0^\infty \frac{us}{(a^2 + u)^3} \frac{du}{2s} \quad (59)$$

$$= \frac{1}{2} \int_0^\infty \frac{u}{(a^2 + u)^3} du \quad (60)$$

$$= \frac{1}{2}(-1) \left[\frac{u}{(u + a^2)^2} + \frac{a^2}{2(a^2 + u)^2} \right]_0^\infty \quad (61)$$

$$= -\frac{1}{2} \left[0 + 0 - 0 - \frac{a^2}{2a^4} \right] \quad (62)$$

$$= \frac{1}{4a^2} \quad (63)$$

The integral over ϕ results in a factor of 2π and the final solution is,

$$\vec{F} = -\frac{q^2}{8\pi^2\epsilon_o}(2\pi)\frac{1}{4a^2}\hat{x} \quad (64)$$

$$= -\frac{q^2}{4\pi\epsilon_o(2a)^2}\hat{x} \quad (65)$$

where this is known to be the correct answer because it represents the force on charge 2 due to the similarly charged particle 1 that is a distance $2a$ away. For the setup I used, the $-\hat{x}$ represents the direction of a repulsive force.

(b) Now we let the particles have opposite charge. We know what answer to expect, the same as in (65) but with opposite sign since these particles will be attracted to each other.

Keep the same setup as in part (a) and use (40) as the electric field due to charge 1. Let charge 2 change to a negative charge. This means the electric field of charge 2 (in the geometry where charge 1 is placed at the origin) is given simply by changing the sign on (42),

$$\vec{E}_2 = -\frac{q}{4\pi\epsilon_o} \left(\frac{1}{a^2 + y^2 + z^2} \right) \left(\frac{a\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{a^2 + y^2 + z^2}} \right) \quad (66)$$

Now comes the big change. The total electric field is completely different from part (a),

$$\vec{E}_{tot} = \vec{E}_1 + \vec{E}_2 \quad (67)$$

$$= \frac{q}{4\pi\epsilon_o(a^2 + y^2 + z^2)^{3/2}}(-2a\hat{x}) \quad (68)$$

because the y and z component terms cancel out due to the new \vec{E}_2 .

Taking into account $E_y = E_z = 0$ and $E_x^2 = E^2$, the stress tensor is,

$$\vec{T} = \epsilon_o \begin{pmatrix} \frac{E^2}{2} & 0 & 0 \\ 0 & -\frac{E^2}{2} & 0 \\ 0 & 0 & -\frac{E^2}{2} \end{pmatrix} \quad (69)$$

The normal vector to the surface enclosing charge 2 is still in the \hat{x} direction because that points outward from the enclosed volume. The dot product of the stress tensor and \hat{x} is,

$$\vec{T} \cdot \hat{x} = \frac{\epsilon_o E^2}{2} \hat{x} \quad (70)$$

$$= \left(\frac{\epsilon_o}{2}\right) \frac{4a^2 q^2}{16\pi^2 \epsilon_o^2 (a^2 + y^2 + z^2)^3} \hat{x} \quad (71)$$

$$= \frac{a^2 q^2}{8\pi^2 \epsilon_o (a^2 + y^2 + z^2)^3} \hat{x} \quad (72)$$

and we should be encouraged to see that so far our solution is at least in the correct direction.

The force is,

$$\vec{F} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{a^2 q^2}{8\pi^2 \epsilon_o (a^2 + y^2 + z^2)^3} dy dz \quad (73)$$

Moving to cylindrical coordinates provides,

$$\vec{F} = \int_0^{\infty} \int_0^{2\pi} \frac{a^2 q^2}{8\pi^2 \epsilon_o (a^2 + s^2)^3} s ds d\phi \hat{x} \quad (74)$$

$$= \frac{a^2 q^2}{4\pi \epsilon_o} \int_0^{\infty} \frac{s}{(a^2 + s^2)^3} ds \hat{x} \quad (75)$$

I use,

$$\int \frac{x}{(f + x^2)^3} dx = -\frac{1}{4(f + x^2)^2} \quad (76)$$

$$\int_0^{\infty} \frac{s}{(a^2 + s^2)^3} ds = -\frac{1}{4(a^2 + s^2)^2} \Big|_0^{\infty} \quad (77)$$

$$= \frac{1}{4a^4} \quad (78)$$

Leading to the solution for the force on charge 2,

$$\vec{F} = \frac{a^2 q^2}{4\pi\epsilon_o} \cdot \frac{1}{4a^4} \hat{x} \quad (79)$$

$$= \frac{q^2}{16\pi\epsilon_o a^2} \hat{x} \quad (80)$$

$$= \frac{q^2}{4\pi\epsilon_o (2a)^2} \hat{x} \quad (81)$$

This is the force we expected all along.

[3.] Problem 8.6 from Griffiths

A parallel plate capacitor is shown in figure 8.6 in Griffiths. We may assume the electric field between the plates of this capacitor is $\vec{E} = E\hat{z}$, and that it is placed in a uniform background magnetic field of $\vec{B} = B\hat{x}$.

(a) Find the EM momentum between the plates.

(b) Connect the plates with a resistive wire stretched along the z -axis. As the capacitor discharges the current in the wire will experience a force due to the background magnetic field. Find the total impulse delivered to the system during the discharge.

(c) Consider an alternative to the situation in part (a). Slowly turn off the magnetic field and let the induced electric field exert a force on the plates. Show that the total impulse delivered to the system is equal to the momentum originally stored in the electromagnetic fields.

Solution

(a) The momentum stored in the EM fields of a particular volume is,

$$\vec{p}_{em} = \mu_o\epsilon_o \int_V \vec{S} d\tau \quad \text{Eq. 8.29} \quad (82)$$

The fields have been given to us and they are uniform. As such, the integral simplifies to the total volume contained within the plates.

$$\vec{p}_{em} = \mu_o\epsilon_o Ad\vec{S} \quad (83)$$

$$= \epsilon_o Ad (E\hat{z} \times B\hat{x}) \quad (84)$$

$$= \epsilon_o AEBd \hat{y} \quad (85)$$

where A is the given area of each plate and d is the distance between them (keep track of d so that you don't mistakenly think it represents a differential element of something).

(b) The impulse delivered to a system is the force applied to it through time.

$$Imp = \int \vec{F} dt \quad (86)$$

For a current flowing in a wire we know the force is $\vec{F} = \vec{I} \times \vec{B}$. The capacitor is discharging so the current is related to the charge on the plates by $I = -dQ/dt$. Putting this together allows for the impulse to be calculated,

$$Imp = \int_0^\infty (\vec{I} \times \vec{B}) dt \quad (87)$$

$$= \int_0^\infty I(d\hat{z} \times B\hat{x}) dt \quad (88)$$

$$= Bd \int_0^\infty \left(-\frac{dQ}{dt} \right) dt \hat{y} \quad (89)$$

$$= -Bd(Q)_0^\infty \hat{y} \quad (90)$$

$$= BdQ_o \hat{y} \quad (91)$$

where $Q = Q(t)$.

The final charge on the plates is zero because eventually the capacitor is completely discharged. The initial charge on the plates can be determined.

$$Q_o = \sigma \cdot A \quad (92)$$

$$= \epsilon_o EA \quad (93)$$

The charge density on the plates for a parallel plate capacitor is known from the relation $E = \sigma/\epsilon_o$.

The total impulse delivered to the capacitor is,

$$Imp = \epsilon_o AEBd \hat{y} \quad (94)$$

which is equivalent to the total momentum originally stored in the fields.

(c) We need to determine the force for which to take our time integral. The problem statement tells us to solve for the induced electric fields so we expect the force to be,

$$\vec{F} = q\vec{E}^* = \epsilon_o EA\vec{E}^* \quad (95)$$

where E is the field given in the problem and \vec{E}^* is the induced field as the magnetic field turns off. This is the expression for the force on one of the plates. One could simply multiply this value by 2 to account for the total force on the capacitor (intuitively one expects that the electric fields will be directed oppositely on the plates, but their charge is also opposite so they experience a force in the same direction). In the following I will leave everything general; skipping over this ability to simplify the problem.

Solve for the induced electric field using Faraday's law. The closed loop is a rectangle of length l and height d . This lengths of this loop lie on the plates and therefore will provide

the induced electric field that exerts a force on them.

$$\oint \vec{E}^* \cdot d\vec{l} = -\frac{d\Phi}{dt} \quad (96)$$

$$E^*(2l) = -ld\frac{dB}{dt} \quad (97)$$

$$E^* = \frac{d}{2} \frac{dB}{dt} \quad (98)$$

The vector nature of E^* is neglected here because the geometry shows that it will be directed in the $\pm\hat{y}$ depending on which plate we consider.

In the Griffiths' figure the bottom plate is shown to have the positive charge density (the electric field points outward from this plate). The induced electric field is directed to the $+\hat{y}$ on this plate. Writing out the total force on this capacitor as the sum of the forces on each plate,

$$\vec{F} = \vec{F}_{bot} + \vec{F}_{top} \quad (99)$$

$$= -\epsilon_o EA \left(\frac{d}{2} \frac{dB}{dt} \hat{y} + \frac{-d}{2} \frac{dB}{dt} (-\hat{y}) \right) \quad (100)$$

$$= -\epsilon_o EAd \frac{dB}{dt} \hat{y} \quad (101)$$

where the initial negative sign comes from Faraday's law.

Now the impulse may be calculated,

$$Imp = \int_0^\infty -\epsilon_o EAd \frac{dB}{dt} dt \hat{y} \quad (102)$$

$$= -\epsilon_o EAd(-B_o) \hat{y} \quad (103)$$

$$= \epsilon_o EABd \hat{y} \quad (104)$$

because the initial magnetic field is given in the problem. This impulse correctly agrees with the original momentum stored in the fields, as given by (85).

[4.] Problem by Professor Carter

Consider a cylindrical capacitor of length L with charge $+Q$ on the inner cylinder of radius a and $-Q$ on the outer cylinder of radius b . The capacitor is filled with a lossless dielectric with dielectric constant equal to 1. The capacitor is located in a region with a uniform magnetic field B , which points along the symmetry axis of the cylindrical capacitor. A flaw develops in the dielectric insulator, and a current flow develops between the two plates of the capacitor. Because of the magnetic field, this current flow results in a torque on the capacitor, which begins to rotate. After the capacitor is fully discharged (total charge on both plates is no zero), what is the magnitude and direction of the angular velocity of the capacitor? The moment of inertia of the capacitor (about the axis of symmetry) is I , and you may ignore fringing fields in the calculation.

Reference the diagram provided on the original homework assignment sheet.

Solution

Considering only the conservation of angular momentum this problem is solved quickly. Before the current flow there is a certain amount of angular momentum stored in the EM fields. After the dielectric breaks down the capacitor discharges until there is no longer an electric field between its plates. Since there is no longer an electric field there is no longer any angular momentum stored in the fields. All of this angular momentum must exist in the physical rotation of the system.

One could solve this problem considering the force on the current, but that would require considerably more work than the method shown here. Furthermore, the problem statement doesn't say much about the current; is it uniform or localized?

The total angular momentum in the fields is,

$$\vec{L}_{em} = \int \epsilon_o \left[\vec{r} \times (\vec{E} \times \vec{B}) \right] d\tau \quad (105)$$

where the integrand is the angular momentum density given as equation 8.34 in Griffiths.

The magnetic field is given in the problem. For a cylindrical capacitor the electric field is known to be zero outside the plates. Inside the capacitor we have,

$$\vec{E} = \frac{\lambda}{2\pi\epsilon s} \hat{s} \quad (106)$$

where λ is the charge per unit length of the inner cylinder and ϵ is the dielectric constant of the material between the plates. We are given that $\epsilon = 1$.

The term $\vec{E} \times \vec{B} = 0$ everywhere except the region between the plates. The limits of the volume integral in (105) are then decided. Continuing with the solution for the total angular momentum stored in the fields gives,

$$\vec{L}_{em} = \int_a^b \int_0^{2\pi} \int_0^L \left[\vec{s} \times \left(\frac{\lambda}{2\pi s} \hat{s} \times B \hat{z} \right) \right] s ds d\phi dz \quad (107)$$

$$= \int_a^b \int_0^{2\pi} \int_0^L \left[s \hat{s} \times \frac{\lambda B}{2\pi s} (-\hat{\phi}) \right] s ds d\phi dz \quad (108)$$

$$= \int_a^b \int_0^{2\pi} \int_0^L -\frac{\lambda B s}{2\pi} ds d\phi dz \hat{z} \quad (109)$$

$$= -\frac{\lambda B}{2\pi} (2\pi)(L) \int_a^b s ds \hat{z} \quad (110)$$

$$= -\frac{\lambda B}{2} (b^2 - a^2) L \hat{z} \quad (111)$$

$$= -\frac{QB}{2} (b^2 - a^2) \hat{z} \quad (112)$$

This is the total angular momentum in the system. When the electric field between the plates of the capacitor goes to zero this angular momentum will be entirely contained within the physical rotation of the system. Angular momentum is related to angular velocity as $\vec{L} = I\vec{\omega}$, where I is the moment of inertia. The solution is,

$$\vec{\omega} = \frac{\vec{L}}{I} = -\frac{QB}{2I}(b^2 - a^2)\hat{z} \quad (113)$$

where this provides both the direction and magnitude of the angular velocity.

[5.] Problem 8.9 from Griffiths

Consider a very long solenoid. This solenoid has radius a , turns per unit length n , and a current I_s flowing through it. A loop of wire with resistance R is coaxial with the solenoid. The radius of this wire loop, b , is much greater than the radius of the solenoid. The current in the solenoid is then slowly decreased, leading to a current flow, I_r , in the loop.

(a) Find I_r in terms of dI_s/dt .

(b) The energy dissipated in the resistor must come from the solenoid. Calculate the Poynting vector just outside the solenoid and verify that it is directed toward the loop. (Griffiths Hint: Use the electric field due to the changing flux in the solenoid and the magnetic field due to the current in the wire loop.) Integrate the Poynting vector over the entire surface of the solenoid to verify that the total energy “emitted” by the solenoid is equal to that dissipated in the resistive wire, $I_r^2 R$.

Solution

(a) This part is a review of the topics covered in Ch. 7 of Griffiths. The current through a resistive wire is $I_r = \mathcal{E}/R$, where \mathcal{E} is the emf (voltage) across the wire. The emf is calculated according to,

$$\mathcal{E} = -\frac{d\Phi}{dt} \quad (114)$$

$$= -\frac{d}{dt}\mu_0 n I_s \pi a^2 \quad (115)$$

$$= -\mu_0 n \pi a^2 \frac{dI_s}{dt} \quad (116)$$

where this takes advantage of the properties of solenoid fields (i.e. the field outside is zero and the field inside is uniform).

The current in the loop is positive, though the direction depends on how we orient the solenoid, let $\vec{B} \rightarrow \hat{z}$.

$$I_r = \frac{\mu_0 n \pi a^2}{R} \left| \frac{dI_s}{dt} \right| \quad (117)$$

The absolute value of the time derivative term is taken because we know the current in the solenoid is decreasing.

(b) Begin by solving for the fields Griffiths tells us to use. The electric field at the surface of the solenoid is found using Faraday’s law, (96). Since we care about the field at the surface

of the solenoid we set $s = a$.

$$E_\phi(2\pi a) = -\frac{d\Phi}{dt} \quad (118)$$

$$\vec{E} = -\mu_o n \pi a^2 \frac{dI_s}{dt} \frac{1}{2\pi a} \hat{\phi} \quad (119)$$

$$= \frac{\mu_o n a}{2} \left| \frac{dI_s}{dt} \right| \hat{\phi} \quad (120)$$

Notice that the final direction of the electric field is in the positive $\hat{\phi}$. There is a negative sign in Faraday's law, but we also know that the time derivative of the solenoid current is negative since it is being slowly decreased.

The magnetic field at the surface of the solenoid is greatly simplified since $b \gg a$. This means we may treat the entire surface as though it lies along the z -axis of the loop. The magnetic field along the z -axis of a current loop is given in example 5.6 of Griffiths,

$$\vec{B}(z) = \frac{\mu_o I_r}{2} \frac{b^2}{(b^2 + z^2)^{3/2}} \hat{z} \quad \text{Eq. 5.38} \quad (121)$$

where the direction is set by the orientation of the solenoid.

The Poynting vector may now be calculated,

$$\vec{S} = \frac{1}{\mu_o} \vec{E} \times \vec{B} = \frac{1}{\mu_o} \frac{\mu_o n a}{2} \left| \frac{dI_s}{dt} \right| \hat{\phi} \times \frac{\mu_o I_r}{2} \frac{b^2}{(b^2 + z^2)^{3/2}} \hat{z} \quad (122)$$

$$= \frac{\mu_o n a b^2 I_r}{4(b^2 + z^2)^{3/2}} \left| \frac{dI_s}{dt} \right| \hat{s} \quad (123)$$

The Poynting vector is directed toward the resistive loop, as expected. The next step is to integrate this vector over the entire surface of the solenoid (still using $s = a$).

$$\text{Power} = \int \vec{S} \cdot d\vec{a} = \int_0^{2\pi} \int_{-\infty}^{\infty} \left[\frac{\mu_o n a b^2 I_r}{4(b^2 + z^2)^{3/2}} \left| \frac{dI_s}{dt} \right| \hat{s} \right] \cdot a d\phi dz \hat{s} \quad (124)$$

$$= \frac{\mu_o n a^2 b^2 I_r}{4} \left| \frac{dI_s}{dt} \right| (2\pi) \int_{-\infty}^{\infty} \frac{dz}{(b^2 + z^2)^{3/2}} \quad (125)$$

From integral tables,

$$\int \frac{dx}{(f + cx^2)^{3/2}} = \frac{x}{f(f + gx^2)^{1/2}} \quad (126)$$

$$\text{Power} = \frac{\mu_o \pi n a^2 b^2 I_r}{2} \left| \frac{dI_s}{dt} \right| \left[\frac{z}{b^2(b^2 + z^2)^{1/2}} \right]_{-\infty}^{\infty} \quad (127)$$

$$= \frac{\mu_o \pi n a^2 b^2 I_r}{2} \left| \frac{dI_s}{dt} \right| \left(\frac{2}{b^2} \right) \quad (128)$$

$$= \mu_o \pi n a^2 I_r \left| \frac{dI_s}{dt} \right| \quad (129)$$

Using (117) to rewrite (129),

$$Power = I_r^2 R \quad (130)$$

and the power directed from the solenoid toward the resistive loop is equal to the energy dissipated in this loop.

[6.] Problem 8.11 from Griffiths

Treat the electron as a uniformly charged spherical shell (charge e) of radius R , spinning with angular velocity ω .

(a) Determine the total energy contained in the EM fields.

(b) Find the total angular momentum contained in the EM fields.

(c) Let the mass of the electron be described completely in terms of the energy stored in its fields, $U_{em} = m_e c^2$. Furthermore, let the spin angular momentum of the electron be due entirely to its fields, $L_{em} = \hbar/2$. Solve for the angular velocity and radius of the electron in this case. Calculate the value $R\omega$ and comment on whether this value makes sense classically.

Solution

(a) The total energy in the fields is given by (21). This is simply the addition of the energy in the electric field plus that of the magnetic field, so I will determine U_e and U_m independently. Begin by determining the fields inside and outside the shell.

From electrostatics and Ch. 2 of Griffiths we know that the electric field inside a uniformly charged spherical shell is zero. Also, the field outside is simply that of a point charge. Therefore,

$$\vec{E}_{in} = 0 \quad (131)$$

$$\vec{E}_{out} = \frac{e}{4\pi\epsilon_0 r^2} \hat{r} \quad (132)$$

The magnetic field inside a uniformly charges spinning spherical shell is given in example 5.11 of Griffiths,

$$\vec{B}_{in} = \frac{2}{3}\mu_0\sigma R\omega \hat{z} \quad \text{Eq. 5.68} \quad (133)$$

This was a remarkable result when we first saw it because the field inside the shell is uniform. This can be written in terms of the variables given in the problem because we can solve for the surface charge density,

$$\sigma = \frac{Q_{tot}}{A} = \frac{e}{4\pi R^2} \quad (134)$$

The magnetic field inside the shell is,

$$\vec{B}_{in} = \frac{\mu_0\omega e}{6\pi R} \hat{z} \quad (135)$$

The magnetic field outside of the shell is that of a dipole. This is known from a variety of sources: HW #9 of PHYS 110A with Prof. Carter, reading problem 5.36 from Griffiths (that

was on HW #9), or using the given vector potential from example 5.11 in Griffiths to solve for the field directly by way of $\vec{B} = \vec{\nabla} \times \vec{A}$. Regardless of your method, the magnetic field outside the shell is,

$$\vec{B}_{out} = \frac{\mu_0 m}{4\pi r^3} (2 \cos \theta \hat{r} + \sin \theta \hat{\theta}) \quad (136)$$

$$= \frac{\mu_0 \omega e R^2}{12\pi r^3} (2 \cos \theta \hat{r} + \sin \theta \hat{\theta}) \quad (137)$$

The magnetic dipole moment was found when this spinning shell was considered in HW #9 mentioned previously. Below is a copy of one of the methods used to find m in that assignment.

Copied from HW #9 Solutions, PHYS 110A Winter 2003

Break the spinning shell into a series of infinitesimal rings. In this case the differential element of the magnetic dipole moment is given by,

$$d\vec{m} = dI \vec{a} \quad (138)$$

The differential element on the right hand side of (138) must be for the current because the area vector of any individual ring is,

$$\begin{aligned} \vec{a} &= \pi l^2 \hat{z} \\ &= \pi (r \sin \theta)^2 \hat{z} \\ \vec{a} &= \pi r^2 \sin^2 \theta \hat{z} \end{aligned} \quad (139)$$

where l is the radius of the ring. The direction \hat{z} is determined by the orientation of the sphere and its rotation and can therefore be set to whatever value we want.

To find dI we need to write out the current in an individual ring. This is determined using the surface current density.

$$\begin{aligned} dI &= K dL \quad \text{where } dL \text{ is the } \hat{\theta} \text{ component of the spherical length } dl \\ &= \sigma v (r d\theta) \\ &= \sigma |\vec{\omega} \times \vec{r}| r d\theta \\ &= \sigma \omega r^2 \sin \theta d\theta \end{aligned}$$

$$d\vec{m} = \pi \sigma \omega R^4 \sin^3 \theta d\theta \quad (140)$$

where I have taken into account the fact that this is a shell so $r = R$.

The magnetic dipole moment is found through an integration of $d\vec{m}$,

$$\begin{aligned}\vec{m} &= \pi\sigma\omega R^4 \int_0^\pi \sin^3\theta \, d\theta \, \hat{z} \\ &= \pi\sigma\omega R^4 \left[-\frac{1}{3} \cos\theta (\sin^2\theta + 2) \right]_0^\pi \hat{z} \\ \vec{m} &= \frac{4\pi\sigma\omega R^4}{3} \hat{z}\end{aligned}\tag{141}$$

End copied section

Replace the charge density in (141) with that from (134) to get the final expression for \vec{B}_{out} . Now begins the calculation of the total energy in the fields. The electric field is zero inside the shell so it contributes nothing to the total energy. The electric field outside the shell contributes,

$$U_{e,out} = \int \frac{\epsilon_o E^2}{2} \, d\tau \tag{142}$$

$$= \frac{\epsilon_o}{2} \int_R^\infty \int_0^\pi \int_0^{2\pi} \left(\frac{e}{4\pi\epsilon_o r^2} \right)^2 r^2 \sin\theta \, dr \, d\theta \, d\phi \tag{143}$$

$$= \frac{e^2}{32\pi^2\epsilon_o} (4\pi) \int_R^\infty \frac{dr}{r^2} \tag{144}$$

$$= \frac{e^2}{8\pi\epsilon_o} \left(-\frac{1}{r} \right)_R^\infty \tag{145}$$

$$= \frac{e^2}{8\pi\epsilon_o R} \tag{146}$$

On to the magnetic field energy inside the shell. Since the field inside the shell is uniform the integral may be skipped. The energy inside the shell is simply the magnetic energy density multiplied by the interior volume.

$$U_{m,in} = u_{m,in} \cdot \frac{4}{3}\pi R^3 \tag{147}$$

$$= \frac{1}{2\mu_o} \left(\frac{\mu_o\omega e}{6\pi R} \right)^2 \cdot \frac{4}{3}\pi R^3 \tag{148}$$

$$= \frac{\mu_o\omega^2 e^2 R}{54\pi} \tag{149}$$

Calculating the energy stored in the magnetic field outside of the shell will illustrate why I

wrote the dipole field in terms of spherical coordinates. Once again, recall that $B^2 = \vec{B} \cdot \vec{B}$.

$$U_{m,out} = \int_R^\infty \int_0^\pi \int_0^{2\pi} \frac{1}{2\mu_o} \left(\frac{\mu_o^2 \omega^2 e^2 R^4}{144\pi^2 r^6} \right) (4 \cos^2 \theta + \sin^2 \theta) r^2 \sin \theta \, dr \, d\theta \, d\phi \quad (150)$$

$$= \frac{\mu_o \omega^2 e^2 R^4}{288\pi^2} (2\pi) \int_R^\infty \int_0^\pi \frac{1}{r^4} (4 \cos^2 \theta + \sin^2 \theta) \sin \theta \, dr \, d\theta \quad (151)$$

$$= \frac{\mu_o \omega^2 e^2 R^4}{144\pi} \left(-\frac{1}{3r^3} \right)_R^\infty \int_0^\pi (4 \cos^2 \theta + \sin^2 \theta) \sin \theta \, d\theta \quad (152)$$

$$= \frac{\mu_o \omega^2 e^2 R}{432\pi} \int_0^\pi (4 \cos^2 \theta + \sin^2 \theta) \sin \theta \, d\theta \quad (153)$$

This is another integral that you may solve in any manner. Here I make the substitution $4 \cos^2 \theta = 4 - 4 \sin^2 \theta$. The integral then reduces to,

$$\int_0^\pi (4 \cos^2 \theta + \sin^2 \theta) \sin \theta \, d\theta = \int_0^\pi (4 - 3 \sin^2 \theta) \sin \theta \, d\theta \quad (154)$$

$$= 4(2) - 3 \int_0^\pi \sin^3 \theta \, d\theta \quad (155)$$

$$= 8 - 3 \left[-\frac{1}{3} \cos \theta (\sin^2 \theta + 2) \right]_0^\pi \quad (156)$$

$$= 8 - 4 = 4 \quad (157)$$

Returning to the energy expression we have,

$$U_{m,out} = \frac{\mu_o \omega^2 e^2 R}{432\pi} \cdot 4 \quad (158)$$

$$= \frac{\mu_o \omega^2 e^2 R}{108\pi} \quad (159)$$

The total energy in the fields is the sum of (146), (149), and (159).

$$U_{tot} = \frac{e^2}{8\pi\epsilon_o R} + \frac{\mu_o \omega^2 e^2 R}{54\pi} + \frac{\mu_o \omega^2 e^2 R}{108\pi} \quad (160)$$

$$= \frac{e^2}{8\pi\epsilon_o R} + \frac{\mu_o \omega^2 e^2 R}{36\pi} \quad (161)$$

(b) The angular momentum stored in the fields is found using (105). The zero electric field inside the shell means that we only need be concerned with the region outside of the shell. All of the fields have already been found, so we begin by finding the angular momentum density in the fields and then we'll integrate that over the volume outside of the shell.

$$\vec{l}_{em} = \epsilon_o \left[\vec{r} \times (\vec{E}_{out} \times \vec{B}_{out}) \right] \quad (162)$$

$$= \epsilon_o \left[\vec{r} \times \left(\frac{e}{4\pi\epsilon_o r^2} \hat{r} \times \frac{\mu_o \omega e R^2}{12\pi r^3} (2 \cos \theta \hat{r} + \sin \theta \hat{\theta}) \right) \right] \quad (163)$$

$$= \epsilon_o \left[\vec{r} \times \left(\frac{e}{4\pi\epsilon_o r^2} \cdot \frac{\mu_o \omega e R^2}{12\pi r^3} \sin \theta \hat{\phi} \right) \right] \quad (164)$$

$$= \frac{\mu_o \omega e^2 R^2}{48\pi^2 r^5} \sin \theta (-r \hat{\theta}) \quad (165)$$

$$= -\frac{\mu_o \omega e^2 R^2}{48\pi^2 r^4} \sin \theta \hat{\theta} \quad (166)$$

On to the total,

$$\vec{L}_{em} = \int_R^\infty \int_0^\pi \int_0^{2\pi} -\frac{\mu_o \omega e^2 R^2}{48\pi^2 r^4} \sin \theta \hat{\theta} r^2 \sin \theta dr d\theta d\phi \quad (167)$$

$$= -\frac{\mu_o \omega e^2 R^2}{48\pi^2} (2\pi) \int_R^\infty \int_0^\pi \frac{\sin^2 \theta}{r^2} dr d\theta \hat{\theta} \quad (168)$$

$$= -\frac{\mu_o \omega e^2 R^2}{24\pi} \left(-\frac{1}{r} \right)_R^\infty \int_0^\pi \sin^2 \theta d\theta \hat{\theta} \quad (169)$$

$$= -\frac{\mu_o \omega e^2 R}{24\pi} \int_0^\pi \sin^2 \theta d\theta \hat{\theta} \quad (170)$$

The integral in (170) is not as easy as it looks. The $\hat{\theta}$ vector changes with the value of θ . Integrating over θ from 0 to π means the $\hat{\theta}$ can be replaced with its z component. This was shown graphically in Discussion on 7-5-2004, but it is a mathematical fact and you will be able to find it elsewhere (proven rigorously). The z component of $\hat{\theta}$ is $-\sin \theta$.

$$-\frac{\mu_o \omega e^2 R}{24\pi} \int_0^\pi \sin^2 \theta d\theta \hat{\theta} = -\frac{\mu_o \omega e^2 R}{24\pi} \int_0^\pi \sin^2 \theta d\theta (-\sin \theta \hat{z}) \quad (171)$$

$$= \frac{\mu_o \omega e^2 R}{24\pi} \int_0^\pi \sin^3 \theta d\theta \hat{z} \quad (172)$$

The solution to the integral is shown in (155) and (156).

$$\vec{L}_{em} = \frac{\mu_o \omega e^2 R}{18\pi} \hat{z} \quad (173)$$

(c) We can use the angular momentum relation to solve for the product, ωR , immediately.

$$L_{em} = \frac{\hbar}{2} \quad (174)$$

$$\frac{\mu_o \omega e^2 R}{18\pi} = \frac{\hbar}{2} \quad (175)$$

$$\omega R = \frac{18\pi \hbar}{2\mu_o e^2} \quad (176)$$

$$= \frac{9\pi(1.05 \times 10^{-34})}{(4\pi \times 10^{-7})(1.60 \times 10^{-19})^2} \quad (177)$$

$$= 9.23 \times 10^{10} \quad (178)$$

This product represents the physical speed of a point on the equator of the shell. It is considerably faster than the speed of light and therefore makes no sense physically, demonstrating the need for quantum mechanics in the explanation of various properties of the electron.

To solve for the values independently use (178) and the mass relation given in the problem statement. This provides two equations with which you can solve for the two unknowns. The numerical values are approximately:

$$R = 2.96 \times 10^{-11} \text{ m} \quad (179)$$

$$\omega = 3.12 \times 10^{21} \text{ s}^{-1} \quad (180)$$

[7.] Problem 9.6 from Griffiths

(a) Write a revised boundary condition (replacing equation 9.27 from Griffiths) for the case of a tension T applied across two strings connected with a knot of mass m .

(b) Consider the situation where the knot connecting the strings has mass m and the second string is massless. Find the amplitudes and phases of the reflected and transmitted waves.

Solution

(a) Begin with,

$$\left. \frac{\partial f}{\partial z} \right|_{0-} = \left. \frac{\partial f}{\partial z} \right|_{0+} \quad \text{Eq. 9.27} \quad (181)$$

where the + and – subscripts refer to the right and left sides of the knot respectively.

To determine the new boundary condition, refer to the origin of 9.27 on page 365 of Griffiths (also presented in lecture on 10-5-2004),

$$\Delta F \cong T \left(\left. \frac{\partial f}{\partial z} \right|_{+} - \left. \frac{\partial f}{\partial z} \right|_{-} \right) \quad (182)$$

The above equation is considered at the point $z = 0$ in the case of a knot being there. Griffiths arrives at equation 9.27 by taking the left side of (182) to be zero because the mass of the knot is set to zero. In part (a) of this problem we are asked to consider a knot of

some mass, m . This is equivalent to setting $\Delta F = ma$, which for a one dimensional case becomes,

$$m \frac{\partial^2 f}{\partial t^2} = T \left(\left. \frac{\partial f}{\partial z} \right|_+ - \left. \frac{\partial f}{\partial z} \right|_- \right) \quad (183)$$

all of which is evaluated at $z = 0$.

(b) This part is a boundary value problem. We will use two equations to solve for the amplitudes and phases of the reflected and transmitted waves in terms of the incident values (which may always be assumed to be given). The first boundary condition is solved for in (183). The second boundary condition comes from the fact that the rope itself is continuous and therefore requires that the functions to the left and right of the knot be equal at $z = 0$.

$$f(0^-, t) = f(0^+, t) \quad \text{Eq. 9.26} \quad (184)$$

The general solution is already known from the properties of waves.

$$f_- = \tilde{A}_I e^{i(k_1 z - \omega t)} + \tilde{A}_R e^{i(-k_1 z - \omega t)} \quad (185)$$

$$f_+ = \tilde{A}_T e^{i(k_2 z - \omega t)} \quad (186)$$

where \tilde{A}_I , \tilde{A}_R , and \tilde{A}_T refer to the complex amplitudes of the incident (coming from the left), reflected, and transmitted waves respectively.

Condition (184) says that we can use either wave function for the time derivative term in (183). Using f_+ we have,

$$m \frac{\partial^2 f}{\partial t^2} = -m\omega^2 \tilde{A}_T e^{-i\omega t} \quad (187)$$

recalling that this is evaluated at $z = 0$. The shortcut method is to use $\partial/\partial t = -i\omega$ when dealing with waves of this sort.

Computing the right side of (183),

$$-m\omega^2 \tilde{A}_T e^{-i\omega t} = iT(k_2 \tilde{A}_T e^{-i\omega t} - k_1 \tilde{A}_I e^{-i\omega t} + k_1 \tilde{A}_R e^{-i\omega t}) \quad (188)$$

$$-m\omega^2 \tilde{A}_T = iT(k_2 \tilde{A}_T - k_1 \tilde{A}_I + k_1 \tilde{A}_R) \quad (189)$$

This allows the transmitted amplitude to be written in terms of the other amplitudes as,

$$(iT k_2 + m\omega^2) \tilde{A}_T = iT k_1 (\tilde{A}_I - \tilde{A}_R) \quad (190)$$

Writing out (184) allows us to simplify it,

$$\tilde{A}_I + \tilde{A}_R = \tilde{A}_T \quad (191)$$

Multiply (191) by iTk_1 and add this to (190),

$$iTk_1 \cdot (\tilde{A}_I + \tilde{A}_R = \tilde{A}_T) \quad (192)$$

+

$$(iTk_2 + m\omega^2) \tilde{A}_T = iTk_1(\tilde{A}_I - \tilde{A}_R) \quad (193)$$

Result :

$$2iTk_1 \tilde{A}_I = [iT(k_1 + k_2) + m\omega^2] \tilde{A}_T \quad (194)$$

$$\tilde{A}_T = \frac{2iTk_1}{[iT(k_1 + k_2) + m\omega^2]} \tilde{A}_I \quad (195)$$

Putting this expression for \tilde{A}_T into (191) and solving for \tilde{A}_R gives,

$$\tilde{A}_R = \frac{iT(k_1 - k_2) - m\omega^2}{iT(k_1 + k_2) + m\omega^2} \tilde{A}_I \quad (196)$$

Now it is time to make use of the fact that the second string is massless. For waves on strings the velocity is given by

$$v = \sqrt{\frac{T}{\mu}} \quad \text{Eq. 9.3} \quad (197)$$

where μ is the mass density of the string.

In this problem $\mu_2 = 0$ so $v_2 = \infty$. The wave vectors of the two waves are related to the velocities by,

$$\frac{k_2}{k_1} = \frac{v_1}{v_2} \quad \text{Eq. 9.24} \quad (198)$$

which leads to $k_2/k_1 = 0$ in this case.

Return to the expressions for the transmitted and reflected amplitudes in terms of the incident amplitude and factor k_1 out of the denominators. This allows those expressions to simplify to,

$$\tilde{A}_T = \frac{2}{1 - \frac{im\omega^2}{k_1 T}} \tilde{A}_I \quad (199)$$

$$\tilde{A}_R = \frac{1 + \frac{im\omega^2}{k_1 T}}{1 - \frac{im\omega^2}{k_1 T}} \tilde{A}_I \quad (200)$$

Separate the real amplitude from the phase of the waves as follows,

$$A_T e^{i\delta_T} = \frac{2}{1 - \frac{im\omega^2}{k_1 T}} A_I e^{i\delta_I} \quad (201)$$

$$A_R e^{i\delta_R} = \frac{1 + \frac{im\omega^2}{k_1 T}}{1 - \frac{im\omega^2}{k_1 T}} A_I e^{i\delta_I} \quad (202)$$

Thus concludes the setup part of this problem. From this point forward it is all algebra. One method is the following,

$$\frac{A_T e^{i\delta_T}}{A_I e^{i\delta_I}} = \frac{2}{1 - \frac{im\omega^2}{k_1 T}} \quad (203)$$

$$\frac{A_T e^{i\delta_T}}{A_R e^{i\delta_R}} = \frac{1 + \frac{im\omega^2}{k_1 T}}{1 - \frac{im\omega^2}{k_1 T}} \quad (204)$$

If you square both sides of the above equations you will be able to separate out the ratio of the amplitudes from a relation between the phases. Coupling this with the following identity,

$$\tan \phi = \frac{e^{i2\phi} - 1}{i(e^{i2\phi} + 1)} \quad (205)$$

will allow you to solve for the desired values in terms of the incident parameters.

After some algebra, the real amplitudes and phases are,

$$A_T = \frac{2}{\sqrt{1 + \frac{m^2\omega^4}{k_1^2 T^2}}} A_I \quad (206)$$

$$A_R = A_I \quad (207)$$

$$\delta_T = \delta_I + \tan^{-1} \frac{m\omega^2}{k_1 T} \quad (208)$$

$$\delta_R = \delta_I + \tan^{-1} \frac{\frac{2m\omega^2}{k_1 T}}{1 - \frac{m^2\omega^4}{k_1^2 T^2}} \quad (209)$$