PHYS 110B - HW #1
Spring 2004, Solutions by David Pace
Any referenced equations are from Griffiths
Problem statements are paraphrased

[1.] Problem 6.8 from Griffiths

A long cylinder has radius R and a magnetization given by $\vec{M} = ks^2\hat{\phi}$. Figure 6.13 in Griffiths represents this cylinder and k is a constant. For points inside and outside the cylinder find the magnetic field due to $\vec{M}$.

Solution

The magnetic field due to $\vec{M}$ is that resulting from the bound currents that $\vec{M}$ generates. These currents are related to the magnetization by,

$$\vec{J}_b = \vec{\nabla} \times \vec{M} \qquad \text{Eq. 6.13} \qquad\qquad \vec{K}_b = \vec{M} \times \hat{n} \qquad \text{Eq. 6.14} \tag{1}$$

It should be immediately noted that $\vec{J}_b = 0$ outside of the cylinder (i.e. for $s > R$) because the magnetization is zero in that region. The bound volume current inside the cylinder may be found using the expression for curl in cylindrical coordinates. All of the terms except one are zero.

$$\vec{J}_b = \vec{\nabla} \times ks^2\hat{\phi} \tag{2}$$

$$= \frac{1}{s}\left[\frac{\partial}{\partial s}(s \cdot ks^2)\right]\hat{z} \tag{3}$$

$$= \frac{1}{s}(3ks^2)\hat{z} \tag{4}$$

$$= 3ks\hat{z} \tag{5}$$

To find the bound surface current use $\hat{n} = \hat{s}$. The problem describes this as a long cylinder, which means that end effects may be neglected. This surface current is only defined at the surface where $s = R$.

$$\vec{K}_b = kR^2\hat{\phi} \times \hat{s} \tag{6}$$

$$= -kR^2\hat{z} \tag{7}$$

since $\hat{\phi} \times \hat{s} = -\hat{z}$.

All of the current is directed along the axis of the cylinder so this problem exhibits symmetry. Ampere's law may be used to find the magnetic field.

$$\oint \vec{B} \cdot d\vec{l} = \mu_o I_{enc} \tag{8}$$

where I_{enc} includes both free and bound currents. The left side of (8) will always be equal to $B_\phi(2\pi s)$ because the magnetic field must always be in the $\hat{\phi}$ direction. This problem reduces to the simple problem of finding the magnetic field a certain distance, s, away from a current-carrying wire.

For the region inside the cylinder, $s < R$, the enclosed current is only that due to the bound volume current density.

$$\begin{aligned} I_{enc} &= \int_0^s \vec{J}_b \cdot d\vec{a} && (9) \\ &= \int_0^s 3ks\hat{z} \cdot s\ ds\ d\phi && (10) \\ &= 3k(2\pi)\frac{s^3}{3} = 2\pi ks^3 && (11) \end{aligned}$$

Using (8) and (11) the magnetic field inside the cylinder is,

$$\vec{B}_{s<R} = \mu_o ks^2\ \hat{\phi} \tag{12}$$

The total enclosed current for the region outside the cylinder, $s > R$, considers the bound volume and bound surface current contributions. The current on the surface is given by the surface current density times the circumference of the cylinder, l.

$$\begin{aligned} I_{enc} &= \int_0^R \vec{J}_b \cdot d\vec{a} + K_b l && (13) \\ &= \int_0^R 3ks \cdot s\ ds\ d\phi + (-kR^2)(2\pi R) && (14) \\ &= 3k(2\pi)\frac{R^3}{3} - 2\pi kR^3 && (15) \\ &= 0 && (16) \end{aligned}$$

Since the total enclosed current is zero, the magnetic field is also zero.

$$\vec{B}_{s>R} = 0 \tag{17}$$

[2.] Problem 6.9 from Griffiths

A short cylinder of length, L, and radius, a, features a permanent magnetization of $\vec{M}$. This magnetization is parallel to the cylinder axis. Find the bound currents and sketch the magnetic field of this cylinder for the geometries where $L \ll a$, $L \gg a$, and $L \approx a$.

Solution

Being uniform and parallel to the cylinder's axis means that the magnetization may be written as,

$$\vec{M} = M_o\hat{z} \tag{18}$$

where M_o is a constant. The bound currents may be found using (1),

$$\begin{aligned} \vec{J}_b &= \vec{\nabla} \times M_o\hat{z} &= 0 && (19) \\ \vec{K}_b &= M_o\hat{z} \times \hat{s} &= M_o\hat{\phi} && (20) \end{aligned}$$

where $\hat{n} = \hat{s}$ for the surface current density because the relevant surface is the side of the cylinder.

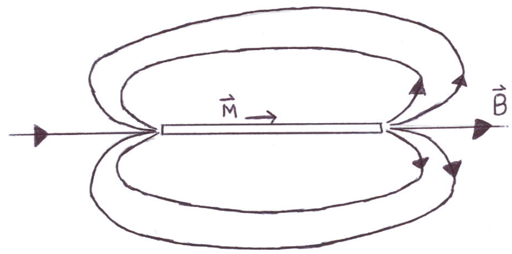

Figure 1: $L \gg a$: Although not clearly indicated by the drawing, the magnetic field inside the cylinder is aligned with the magnetization that is shown. This field is uniform inside the cylinder.

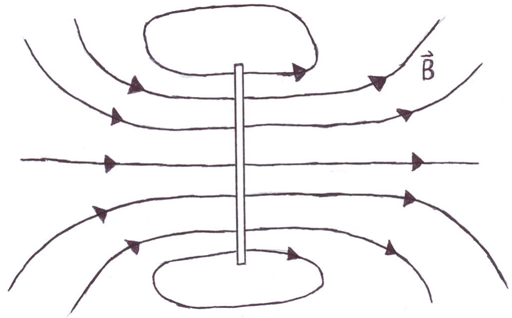

Figure 2: $L \ll a$: In this case the cylinder looks like a single loop of current. The field in this configuration is essentially that of a physical magnetic dipole.

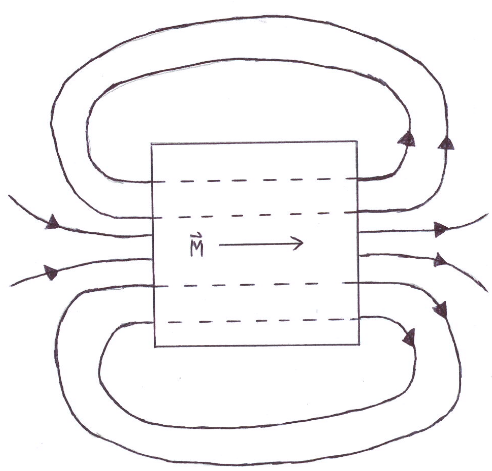

Figure 3: $L \approx a$: The magnetic field lines outside the cylinder (which are not labelled) once again match those of a dipole. Inside the cylinder the magnetization is aligned with the magnetic field (dashed lines).

[3.] Problem 6.12 from Griffiths

An infinitely long cylinder has a magnetization given by, $\vec{M} = ks\hat{z}$. This cylinder has radius, R, and k is a constant while s is the radial coordinate in the cylindrical system. There is no free current anywhere. Find the magnetic field inside and outside of this cylinder by the following methods:

(a) Locate all of the bound currents and determine the field from them.

(b) Use the form of Ampere's Law that includes $\vec{H}$ to find the field.

Solution

(a) The bound currents may once again be found from (1),

$$\vec{J}_b = \vec{\nabla} \times ks\hat{z} = -\frac{\partial}{\partial s}(ks)\hat{\phi} = -k\hat{\phi} \tag{21}$$

$$\vec{K}_b = kR\hat{z} \times \hat{s} = kR\hat{\phi} \tag{22}$$

where only the non-zero parts of the curl calculation are shown, and $s = R$ for the bound surface current calculation.

These currents will be used with Ampere's Law, (8), to determine the magnetic field. Both bound currents are directed along $\hat{\phi}$, and therefore we apply Ampere's Law as was done for a solenoid in example 5.9 (p. 227) of Griffiths. Referencing that example, figure 5.37 illustrates how with currents directed solely along $\hat{\phi}$ there will be no magnetic field outside the cylinder. Outside of the cylinder then,

$$\vec{B}_{out} = 0 \tag{23}$$

This can also be confirmed by reviewing the solution to problem 5.15 from Griffiths in 110A last quarter. As figure 4 shows, the Amperian loop taken to determine the magnetic field inside the cylinder extends to the region outside the cylinder.

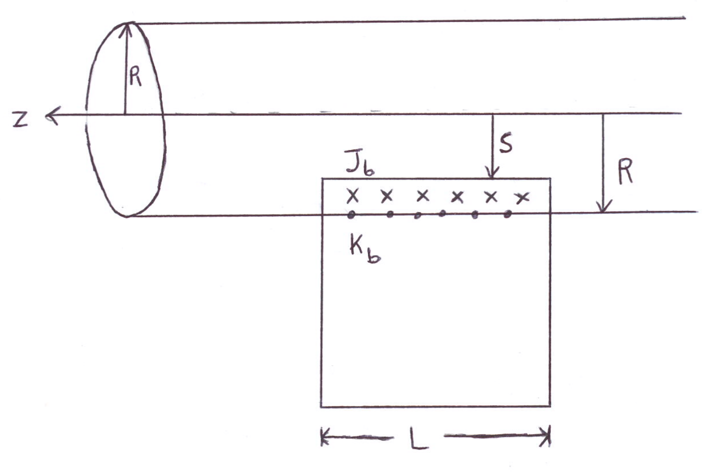

Figure 4: Diagram illustrating the Amperian loop of problem 6.12.

Inside the cylinder we need to determine the total enclosed current. Ampere's Law, (8), simplifies in this case,

$$\oint \vec{B} \cdot d\vec{l} = \mu_o I_{enc} \tag{24}$$

$$BL = \mu_o I_{enc} \tag{25}$$

The enclosed current includes the volume and surface contributions. Note that $d\vec{l}$ points along the $\hat{z}$ direction. Since the bound current densities are both constant there will not be any integrals to determine the total current.

$$I_{enc} = J \cdot \text{area} + K \cdot \text{length} \tag{26}$$

$$= J_b(R-s)L + K_b L \tag{27}$$

$$= -kL(R-s) + kRL \quad = \quad ksL \tag{28}$$

Using (28) and (25) an expression for the magnetic field inside the cylinder can be found,

$$\vec{B}_{in} = \mu_o ks\hat{z} \tag{29}$$

(b) The version of Ampere's Law required for this part is,

$$\oint \vec{H} \cdot d\vec{l} = I_{f_{enc}} \tag{30}$$

The problem makes a point to state that $I_f = 0$. This means that $\vec{H} = 0$ and we should turn to the relation between $\vec{H}$, $\vec{B}$, and $\vec{M}$.

$$\vec{H} \equiv \frac{1}{\mu_o}\vec{B} - \vec{M} \qquad \text{Eq. 6.18} \tag{31}$$

Since $\vec{H} = 0$ we immediately know,

$$\vec{B} = \mu_o \vec{M} \tag{32}$$

where $\vec{M}$ is given in the problem.

The magnetic fields inside and outside the cylinder are,

$$\vec{B}_{out} = 0 \tag{33}$$

$$\vec{B}_{in} = \mu_o ks\hat{z} \tag{34}$$

just as found in part (a).

[4.] Problem 6.15 from Griffiths

Find the field inside the uniformly magnetized sphere of example 6.1 (p. 264). Take the following as a hint: There is no free current in this sphere, therefore, the $\vec{H}$ field may be written as, $\vec{H} = -\vec{\nabla}W$ where W is a scalar potential. By way of equation 6.23 it is also known that, $\nabla^2 W = \vec{\nabla} \cdot \vec{M}$. Finally, $\vec{\nabla} \cdot \vec{M} = 0$ everywhere except at the surface of the sphere.

Solution

This is a problem concerning Laplace's equation, $\nabla^2 W = 0$. It is known that the solutions to Laplace's equation in spherical geometry with azimuthal symmetry (i.e. no ϕ dependence) are of the form,

$$\Xi = \sum_{l}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta) \tag{35}$$

where Ξ is any function satisfying $\nabla^2 \Xi = 0$, the A's and B's are arbitrary constants, and the P_l terms are the Legendre polynomials.

Inside the sphere there cannot be a $1/r$ term because that would represent an infinite W at $r = 0$. Outside the sphere there can be no r term because that would represent an infinite W at $r \to \infty$. Therefore the expressions for W inside and outside the sphere may be partially simplified and written as,

$$W_{in} = \sum_{l}^{\infty} A_l r^l P_l(\cos\theta) \tag{36}$$

$$W_{out} = \sum_{l}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \tag{37}$$

This scalar potential is a continuous function (treat it like V in the previous boundary value problem of electrostatics), therefore we have our first boundary condition at the surface of the sphere,

$$W_{in}(r = R) = W_{out}(r = R) \tag{38}$$

$$\sum_{l}^{\infty} A_l R^l P_l(\cos\theta) = \sum_{l}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) \tag{39}$$

The orthogonality of the Legendre polynomials means that we can equate each l term separately.

$$A_l R^l = \frac{B_l}{R^{l+1}} \tag{40}$$

The problem only asks for the field inside the sphere, but the field inside must still adhere to boundary conditions that relate to the field outside. In (40) we relate the A_l terms (which we want to solve) to the B_l terms that we do not care about. After finding another relation between these variables we can neglect the B_l terms and finish the problem.

There is another condition at the sphere's surface that will provide the second equation we need to solve for the A_l terms.

$$H^{\perp}_{above} - H^{\perp}_{below} = -(M^{\perp}_{above} - M^{\perp}_{below}) \qquad \text{Eq. 6.24} \tag{41}$$

where the *above* and *below* subscripts are equivalent to *outside* and *inside* respectively.

In this spherical geometry the perpendicular components at $r = R$ are those along the $\hat{r}$ direction. Beginning with the right side of (41), $M_{above} = M_{out} = 0$ since there is no magnetization outside of the sphere. This means that the perpendicular component of $\vec{M}$ must also be zero.

From example 6.1 we may write the magnetization of the sphere as $\vec{M} = M_o\hat{z}$, where M_o is a constant. Using $\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$, the $\hat{r}$ component of $\vec{M}_{in}$ is,

$$M^{\perp}_{in} = M_o \cos\theta \tag{42}$$

and this is the right side of (41).

For the left side of (41) we keep only the $\hat{r}$ component of the $\vec{H} = -\vec{\nabla}W$ expression. Taking the expression for the gradient in spherical coordinates we get (recalling that there is already a negative sign in the expression),

$$H^{\perp}_{out} = -\frac{\partial}{\partial r}W_{out} = -\sum_{l}^{\infty}\frac{-(l+1)B_l}{r^{l+2}}P_l(\cos\theta) = \sum_{l}^{\infty}\frac{(l+1)B_l}{R^{l+2}}P_l(\cos\theta) \tag{43}$$

$$H^{\perp}_{in} = -\frac{\partial}{\partial r}W_{in} = -\sum_{l}^{\infty} lA_l r^{l-1}P_l(\cos\theta) = -\sum_{l}^{\infty} lA_l R^{l-1}P_l(\cos\theta) \tag{44}$$

where the last step in each line incorporated the fact that $r = R$.

Taking all of the information just calculated allows us to rewrite (41) as,

$$\sum_{l}^{\infty}\frac{(l+1)B_l}{R^{l+2}}P_l(\cos\theta) - \left(-\sum_{l}^{\infty} lA_l R^{l-1}P_l(\cos\theta)\right) = M_o\cos\theta \tag{45}$$

The right side includes a $\cos\theta$ term, which is equivalent to $P_1(\cos\theta)$. As such, the left side of the above expression must have only $l = 1$ terms. All other values of A_l and B_l are zero. Rewriting (45),

$$\frac{(2)B_1}{R^3}P_1(\cos\theta) + A_1R^0P_1(\cos\theta) = M_oP_1(\cos\theta) \tag{46}$$

$$\frac{2B_1}{R^3} + A_1 = M_o \tag{47}$$

From (40) we have,

$$B_1 = R^3A_1 \tag{48}$$

and this may be put into (47) as follows,

$$\frac{2}{R^3}R^3A_1 + A_1 = M_o \tag{49}$$

$$A_1 = \frac{M_o}{3} \tag{50}$$

The scalar potential inside the sphere is,

$$W_{in} = A_1 r\cos\theta = \frac{M_o}{3} r\cos\theta \tag{51}$$

Now we can solve for $\vec{H}$ inside the sphere,

$$\vec{H}_{in} = -\vec{\nabla}W_{in} = -\left[\frac{\partial}{\partial r}\frac{M_o}{3}r\cos\theta\,\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\frac{M_o}{3}r\cos\theta\,\hat{\theta}\right] \tag{52}$$

$$= -\frac{M_o}{3}(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}) \tag{53}$$

$$= -\frac{M_o}{3}\hat{z} \tag{54}$$

The magnetic field, $\vec{B}$, may be found using (31) and the fact that $\vec{M}_{in} = M_o\hat{z}$,

$$\vec{B}_{in} = \mu_o(\vec{H}_{in} + \vec{M}_{in}) \tag{55}$$

$$= \mu_o\left(-\frac{M_o}{3}\hat{z} + M_o\hat{z}\right) \tag{56}$$

$$= \frac{2}{3}\mu_o M_o\hat{z} \tag{57}$$

$$= \frac{2}{3}\mu_o\vec{M} \tag{58}$$

where (58) is the solution given as Griffiths equation 6.16. The method used here is just another way to solve the problem.

[5.] Modified Version of Problem 6.16 in Griffiths

A coaxial cable consists of a conducting wire of radius a surrounded by a conducting cylindrical tube of radius c. A current I flows down the wire and returns along the outer conductor (see figure 6.24 in Griffiths), in both cases the current is distributed uniformly. Two different magnetic materials exist in the space between the two conductors of the cable: the first, with susceptibility $\chi_{m,1}$, fills the space from $a < r < b$ and the second, with susceptibility $\chi_{m,2}$, fills the space from $b < r < c$.

(a) Find the magnetic field in the region between the two conductors.

(b) Calculate the magnetization and find the bound currents; make sure these are consistent with the answer to part (a).

Solution

Begin by noting that the problem writes the radial coordinate in terms of r, while Griffiths uses s. This solution will use s so that the method better matches that given in the text and used on other problems.

(a) Let the current in the center wire be along the $\hat{z}$ direction. The current in the outer shell must then be along the $-\hat{z}$ direction, but this will not matter since we only need to solve for the fields inside the coaxial cable. The free currents are given in the problem and they exhibit cylindrical symmetry. We can use (30) to solve for $\vec{H}$ and then find $\vec{B}$.

$$H_\phi(2\pi s) \quad = \quad I_f \tag{59}$$

$$\vec{H}_{in} \quad = \quad \frac{I}{2\pi s}\hat{\phi} \tag{60}$$

where $\vec{H}_{in}$ is the field in the region between the conductors. This does not depend on the materials inside the coaxial cable because $\vec{H}$ is determined entirely from free currents.

There is a relation between $\vec{B}$ and $\vec{H}$ fields in linear media (it is safe to assume that all magnetic materials studied in this course will be linear),

$$\vec{B} = \mu_o(1+\chi_m)\vec{H} \qquad \text{Eqs. 6.31 and 6.32} \tag{61}$$

This shows how the materials affect the field. There is a value of $\vec{B}$ for the region occupied by each material. Calling these regions 1 and 2 for containing materials with $\chi_{m,1}$ and $\chi_{m,2}$ respectively,

$$\vec{B}_1 \quad = \quad \mu_o(1+\chi_{m,1})\frac{I}{2\pi s}\hat{\phi} \tag{62}$$

$$\vec{B}_2 \quad = \quad \mu_o(1+\chi_{m,2})\frac{I}{2\pi s}\hat{\phi} \tag{63}$$

(b) In linear media the magnetization is given by,

$$\vec{M} = \chi_m\vec{H} \qquad \text{Eq. 6.29} \tag{64}$$

The magnetization will therefore vary according to which material we are considering. This is similar to the manner in which the polarization in a region of space depends on the dielectric properties of any materials there. The magnetization in each region is,

$$\vec{M}_1 \quad = \quad \chi_{m,1}\frac{I}{2\pi s}\hat{\phi} \tag{65}$$

$$\vec{M}_2 \quad = \quad \chi_{m,2}\frac{I}{2\pi s}\hat{\phi} \tag{66}$$

Once again, the bound current densities are given by (1). Beginning with the volume bound current densities,

$$\vec{J}_{b,1} \quad = \quad \vec{\nabla}\times\frac{\chi_{m,1}I}{2\pi s}\hat{\phi} \quad = \quad \frac{1}{s}\left[\frac{\partial}{\partial s}\left(s\cdot\frac{\chi_{m,1}I}{2\pi s}\right)\right]\hat{z} \quad = \quad 0 \tag{67}$$

$$\vec{J}_{b,2} \quad = \quad \vec{\nabla}\times\frac{\chi_{m,2}I}{2\pi s}\hat{\phi} \quad = \quad \frac{1}{s}\left[\frac{\partial}{\partial s}\left(s\cdot\frac{\chi_{m,2}I}{2\pi s}\right)\right]\hat{z} \quad = \quad 0 \tag{68}$$

There is no bound volume current.

There are three surfaces at which to determine bound surface current density, $s = a$, $s = b$, and $s = c$. This is where the surfaces of the magnetic materials may be found. For $s = a$ only the magnetization of material 1 matters, and the normal vector to the surface is directed along $-\hat{s}$,

$$\vec{K}_b = \frac{\chi_{m,1} I}{2\pi s}\hat{\phi} \times (-\hat{s}) = \frac{\chi_{m,1} I}{2\pi a}\hat{z} \qquad \text{at } s = a \tag{69}$$

At $s = b$ there are two separate bound surface current densities to consider. Each material makes a contribution and the total density on that surface is their sum. The direction of the normal vector is reversed for each region.

$$\vec{K}_{b,1} = \frac{\chi_{m,1} I}{2\pi b}\hat{\phi} \times \hat{s} = -\frac{\chi_{m,1} I}{2\pi b}\hat{z} \tag{70}$$

$$\vec{K}_{b,2} = \frac{\chi_{m,2} I}{2\pi b}\hat{\phi} \times (-\hat{s}) = \frac{\chi_{m,2} I}{2\pi b}\hat{z} \tag{71}$$

The total surface current density at $s = b$ is,

$$\vec{K}_b = \frac{I}{2\pi b}(\chi_{m,2} - \chi_{m,1})\hat{z} \qquad \text{at } s = b \tag{72}$$

At $s = c$ the same method is employed,

$$\vec{K}_b = \frac{\chi_{m,2} I}{2\pi c}\hat{\phi} \times \hat{s} = -\frac{\chi_{m,2} I}{2\pi c}\hat{z} \qquad \text{at } s = c \tag{73}$$

All of the bound currents are directed along the $\hat{z}$ axis (positive or negative). Ampere's Law in the form of (8) can be used to find $\vec{B}$ and make sure it agrees with the value given in part (a). Cylindrical symmetry is maintained, so we have,

$$B_\phi(2\pi s) = \mu_o I_{enc} \tag{74}$$

where I_{enc} is the total enclosed current (bound and free). The free current is a simple constant that is given. The bound current is found from the bound surface current density. This density is constant, so the total bound current on any surface is found by multiplying the current density by the length across the surface. This length is the circumference in our cylindrically symmetric system.

In region 1 our Amperian loop encloses the free current and the current on the surface at $s = a$,

$$I_{enc} = I + K_{b,a} L = I + \frac{\chi_{m,1} I}{2\pi a}(2\pi a) = I + \chi_{m,1} I = (1 + \chi_{m,1}) I \tag{75}$$

The magnetic field in region 1 is,

$$\vec{B}_1 = \mu_o (1 + \chi_{m,1}) \frac{I}{2\pi s}\hat{\phi} \tag{76}$$

which agrees with the solution in part (a).

Region 2 has a different value for the total enclosed current,

$$I_{enc} = I_f + I_{s=a} + I_{s=b} = I + \chi_{m,1} I + \left[(\chi_{m,2} - \chi_{m,1}) \frac{I}{2\pi b} \right] \cdot 2\pi b \qquad (77)$$

$$= I + \chi_{m,1} I + \chi_{m,2} I - \chi_{m,1} I = (1 + \chi_{m,2}) I \qquad (78)$$

The magnetic field in region 2 is,

$$\mu_o (1 + \chi_{m,2}) \frac{I}{2\pi s} \hat{\phi} \qquad (79)$$

which agrees with the solution in part (a).

In this problem the magnetic materials can increase or decrease the field. In linear materials they do not change the direction of the field. Taking this into account it is sensible that the magnetic field inside the cable retains the $1/s$ dependence that we normally see in the field due to a current along a wire.

[6.] Problem 6.17 from Griffiths

A current I flows in a wire of radius a. The wire has susceptibility χ_m. For a uniform current distribution find the magnetic field a distance s away from the wire's axis. Determine all of the bound currents and the net bound current in the wire.

Solution

One way to find the magnetic field in all space is to use Ampere's Law for $\vec{H}$, which is given by (30). This will require us to determine the enclosed current for Amperian loops that may be inside or outside the wire. Since the total free current, I, is said to be uniform we may rewrite it as a free volume current density,

$$\vec{J}_f = \frac{I}{\pi a^2} \hat{z} \qquad (80)$$

where πa^2 is the cross sectional area of the wire and I have set the direction to be along the $\hat{z}$.

Inside the wire where $s < a$, we have,

$$H_\phi (2\pi s) = \frac{I}{\pi a^2} (\pi s^2) \qquad (81)$$

$$\vec{H}_{in} = \frac{sI}{2\pi a^2} \hat{\phi} \qquad (82)$$

The relation between $\vec{B}$ and $\vec{H}$ in a linear magnetic material is given by (61), which we now use to solve for the magnetic field inside the wire,

$$\vec{B}_{in} = \mu_o (1 + \chi_m) \frac{sI}{2\pi a^2} \hat{\phi} \qquad (83)$$

Outside of the wire, where $s > a$, our Amperian loop encloses all of the free current.

$$H_\phi (2\pi s) = I \qquad (84)$$

$$\vec{H}_{out} = \frac{I}{2\pi s} \hat{\phi} \qquad (85)$$

Again we use (61) to solve for the magnetic field in this region where $\vec{H}$ is known,

$$\vec{B}_{out} = \mu_o(1+\chi_{m,out})\frac{I}{2\pi s}\hat{\phi} \tag{86}$$

$$= \frac{\mu_o I}{2\pi s}\hat{\phi} \tag{87}$$

since the susceptibility of vacuum, $\chi_{m,out}$, is zero. This is the magnetic field due to a current carrying wire (outside the wire), which is exactly what the problem gives.

To find the bound currents we turn to (1). Begin with the bound surface current density since that exists only at $s = a$. Recalling that the magnetization in the wire is given by (64),

$$\vec{K}_b = \chi_m \vec{H}_{in} \times \hat{s} = -\frac{\chi_m I}{2\pi a}\hat{z} \tag{88}$$

The presence of linear media allow for the determination of bound volume current densities from,

$$\vec{J}_b = \chi_m \vec{J}_f \qquad \text{Eq. 6.33} \tag{89}$$

Inside the wire then,

$$\vec{J}_b = \frac{\chi_m I}{\pi a^2} \tag{90}$$

This is the short method for finding $\vec{J}_b$, but the original method given in (1) will also work.

$$\vec{J}_b = \vec{\nabla} \times \chi_m \vec{H}_{in} = \vec{\nabla} \times \frac{\chi_m I s}{2\pi a^2}\hat{\phi} \tag{91}$$

$$= \frac{1}{s}\left[\frac{\partial}{\partial s}\left(s \cdot \frac{\chi_m I s}{2\pi a^2}\right)\right]\hat{z} \tag{92}$$

$$= \frac{\chi_m I}{2\pi a^2 s} \cdot 2s\hat{z} \tag{93}$$

$$= \frac{\chi_m I}{\pi a^2}\hat{z} \tag{94}$$

The total bound current may now be calculated. Both the surface and volume bound current densities are constant, so the total current is found from,

$$I_{b,tot} = J_b \cdot \text{area} + K_b \cdot \text{length} \tag{95}$$

$$= \frac{\chi_m I}{\pi a^2}(\pi a^2) + \left(-\frac{\chi_m I}{2\pi a}\right)(2\pi a) = 0 \tag{96}$$

The net bound current is zero. Since these currents are limited to the material (i.e. they are *bound*), this result should be expected. In order for there to be a net bound current, the bound currents would have to be able to leave the object.

[7.] Problem 6.18 from Griffiths

A sphere is placed in a uniform background magnetic field. The sphere is made of linear magnetic material and the background field is $\vec{B}_o$. Find the magnetic field inside the sphere. (Prof. Carter Hint: use the method of problem 6.15, referring to example 4.7)

Solution

Professor Carter solved this problem using an iterative method in lecture (April 9, 2004). Here we will follow the hint and solve this as a separation of variables problem.

There is no free current in the sphere; we know this because free current would only exist in the sphere if it was purposely put there, and then the problem statement would describe it. Without free current the curl of $\vec{H}$ is zero and have (just as problem 6.15 describes),

$$\vec{H} = -\vec{\nabla} W \tag{97}$$

$$\nabla^2 W = 0 \qquad \text{Inside and outside the sphere} \tag{98}$$

where W is again a scalar potential.

The plan is to solve for W inside the sphere, use that value to solve for $\vec{H}_{in}$, and then use $\vec{H}_{in}$ to solve for $\vec{B}_{in}$. Inside the sphere we have,

$$W_{in} = \sum_l^\infty A_l r^l P_l(\cos\theta) \tag{99}$$

which is the given solution for Laplace's equation in spherically symmetric systems.

Before we can write the expression for W outside the sphere we have to determine the value of W as $r \to \infty$. The background field is $\vec{B}_o = B_o \hat{z}$. The scalar potential is in terms of $\vec{H}$ and leads to,

$$W(r \to \infty) = -H_o r\cos\theta = -\frac{B_o}{\mu_o} r\cos\theta \tag{100}$$

where this makes use of $B = \mu_o H$ outside the sphere.

The expression for the potential outside of the sphere contains this term and the general sum,

$$W_{out} = -\frac{B_o}{\mu_o} r\cos\theta + \sum_l^\infty \frac{C_l}{r^{l+1}} P_l(\cos\theta) \tag{101}$$

where C_l is an unknown constant.

At the surface of the sphere, $W_{in}(r = R) = W_{out}(r = R)$.

$$\sum_l^\infty A_l R^l P_l(\cos\theta) = -\frac{B_o}{\mu_o} R\cos\theta + \sum_l^\infty \frac{C_l}{R^{l+1}} P_l(\cos\theta) \tag{102}$$

Noting that $\cos\theta = P_1(\cos\theta)$, (102) gives two separate expressions,

$$A_1 R = -\frac{B_o}{\mu_o} R + \frac{C_1}{R^2} \qquad \text{for } l = 1 \tag{103}$$

$$A_l R^l = \frac{C_l}{R^{l+1}} \qquad \text{for } l \neq 1 \tag{104}$$

In the equations above there are (essentially) two unknowns and one equation. We can use another boundary condition at the surface of the sphere to get another equation. In problem 6.15 we used (41), but in this problem we don't know anything about the magnetization inside the sphere. We do, however, have information about the magnetic field across this boundary and we may consider,

$$B^{\perp}_{above} - B^{\perp}_{below} \quad = \quad 0 \qquad \text{Eq. 6.26} \tag{105}$$

Solving for $B^{\perp}_{above}$, which is evaluated at $r = R$,

$$B^{\perp}_{above} \quad = \quad \mu_o H^{\perp}_{out} \;=\; -\mu_o \frac{\partial}{\partial r} W_{out} \tag{106}$$

$$= \; -\mu_o \left[-\frac{B_o}{\mu_o}\cos\theta + \sum_l^{\infty} \frac{-(l+1)C_l}{R^{l+2}} P_l(\cos\theta) \right] \tag{107}$$

$$= \; B_o P_1(\cos\theta) + \sum_l^{\infty} \frac{\mu_o (l+1) C_l}{R^{l+2}} P_l(\cos\theta) \tag{108}$$

where the perpendicular components are the $\hat{r}$ components of the gradient.

To find $B^{\perp}_{below}$ we follow a similar method, but in this case $B = \mu H$, where $\mu = \mu_o(1 + \chi_m)$. Since the sphere is made of linear magnetic material we know that its susceptibility may be written as χ_m.

$$B^{\perp}_{below} \quad = \quad \mu H^{\perp}_{in} \;=\; -\mu \frac{\partial}{\partial r} W_{in} \tag{109}$$

$$= \; -\sum_l^{\infty} \mu l A_l R^{l-1} P_l(\cos\theta) \tag{110}$$

The expressions (110) and (108) are equal according to the boundary condition given in (105).

$$-B_o - \frac{2\mu_o C_1}{R^3} \quad = \quad \mu_o(1+\chi_m) A_1 \qquad \text{for } l = 1 \tag{111}$$

$$-\frac{(l+1)C_1}{R^{l+2}} \quad = \quad l A_l R^{l-1} \qquad \text{for } l \neq 1 \tag{112}$$

Solving for the $l \neq 1$ terms first, insert (104) into (112),

$$-\frac{(l+1)}{R^{l+2}} \left(A_l R^{2l+1} \right) \quad = \quad l A_l R^{l-1} \tag{113}$$

$$-(l+1) R^{2l+1} \quad = \quad l R^{l-1} R^{l+2} \tag{114}$$

$$-(l+1) R^{2l+1} \quad = \quad l R^{2l+1} \tag{115}$$

$$-l - 1 \quad = \quad l \tag{116}$$

$$l = -\frac{1}{2} \tag{117}$$

so the $l \neq 1$ terms only exist for the $l = -1/2$ value. Our sum for the potential requires that the l's be positive whole numbers, therefore, the only terms that exist in the final solution are the $l = 1$ terms.

Solve for the $l = 1$ terms by inserting (111) into (103),

$$A_1 R = -\frac{B_o}{\mu_o} R + \frac{1}{R^2}\left[-\frac{R^3}{2\mu_o}(\mu_o(1+\chi_m)A_1 + B_o)\right] \tag{118}$$

$$A_1 = -\frac{B_o}{\mu_o} - \frac{1}{2\mu_o}(\mu_o(1+\chi_m)A_1 + B_o) \tag{119}$$

$$A_1\left(1 + \frac{\mu_o(1+\chi_m)}{2\mu_o}\right) = -\frac{3B_o}{2\mu_o} \tag{120}$$

$$A_1 = -\frac{3B_o}{2\mu_o}\left(\frac{2\mu_o}{2\mu_o + \mu_o(1+\chi_m)}\right) \tag{121}$$

$$A_1 = -\frac{3B_o}{\mu_o}\left(\frac{1}{3+\chi_m}\right) \tag{122}$$

For the sake of eventually matching the solution presented in lecture, I will rewrite A_1 as,

$$A_1 = -\frac{B_o}{\mu_o}\left(\frac{1}{1+\frac{\chi_m}{3}}\right) \tag{123}$$

Now that we have A_1 we know the value of W inside the sphere,

$$W_{in} = -\frac{B_o}{\mu_o}\left(\frac{1}{1+\frac{\chi_m}{3}}\right) r\cos\theta \tag{124}$$

The value of $\vec{H}_{in}$ is,

$$\vec{H}_{in} = -\vec{\nabla} W_{in} = -\left[\frac{\partial}{\partial r} W_{in}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta} W_{in}\hat{\theta}\right] \tag{125}$$

$$= -\left[-\frac{B_o}{\mu_o}\left(\frac{1}{1+\frac{\chi_m}{3}}\right)\right]\left(\cos\theta\hat{r} + \frac{1}{r}(r)(-\sin\theta)\hat{\theta}\right) \tag{126}$$

$$= \frac{B_o}{\mu_o}\left(\frac{1}{1+\frac{\chi_m}{3}}\right)\hat{z} \tag{127}$$

Finally, the magnetic field inside the sphere is,

$$\vec{B}_{in} = \mu\vec{H}_{in} = \mu_o(1+\chi_m)\cdot\frac{B_o}{\mu_o}\left(\frac{1}{1+\frac{\chi_m}{3}}\right)\hat{z} \tag{128}$$

$$= \frac{1+\chi_m}{1+\frac{\chi_m}{3}} B_o\,\hat{z} \tag{129}$$

which agrees with the iterative method of lecture.