## PHYS 110A - HW \#5

Solutions by David Pace
Any referenced equations are from Griffiths
[1.] Problem 3.28 from Griffiths Following Example 3.9 of Griffiths (page 142) consider a spherical shell of radius R and surface charge density $\sigma=k \cos \theta$.
(a) Find the dipole moment of this distribution.
(b) Find the approximate potential at points far away from the shell. Compare your solution with the exact answer given as equation 3.87. What does this mean about the higher multipoles?

## Part (a)

The dipole moment of a surface charge is given by,

$$
\vec{p}=\int \overrightarrow{r^{\prime}} \sigma\left(\overrightarrow{r^{\prime}}\right) d a^{\prime}
$$

This is not explicitly given in Griffiths, but equation 3.98 gives the dipole moment for a volume charge distribution and it is said that the equivalent expressions for line and surface charges are similarly found.


Figure 1: Geometry of problem [1.]. The shell is shown as a two dimensional cross section.
Use figure 1 to determine the direction of $\vec{p}$. Dipole moments always point from the negative charge to the positive. The $\cos \theta$ term has positive values in the region above the plane (where $\theta$ ranges from 0 to $\frac{\pi}{2}$ ). This term is negative in the region below the plane. Using these facts and the symmetry of the shell we can determine that the dipole moment must point in the positive $\hat{z}$ direction. I use primes to denote coordinates with respect to the shell.

$$
\begin{aligned}
\vec{p} & =\int \overrightarrow{r^{\prime}} k \cos \theta^{\prime} r^{\prime 2} \sin \theta^{\prime} d \theta^{\prime} d \phi^{\prime} \\
& =\int \overrightarrow{z^{\prime}} k \cos \theta^{\prime} r^{\prime 2} \sin \theta^{\prime} d \theta^{\prime} d \phi^{\prime} \\
& =2 \pi k \int z^{\prime} \cos \theta^{\prime} \sin \theta^{\prime} r^{\prime 2} d \theta^{\prime}
\end{aligned}
$$

This is a surface integral so $\mathrm{r}^{\prime}$ is evaluated at R (the surface of the shell). Also, recall that $z^{\prime}=r^{\prime} \cos \theta^{\prime}$ and that eventually we have to include the limits of our integration.

$$
\begin{aligned}
\vec{p} & =2 \pi k \int r^{\prime 3} \cos ^{2} \theta^{\prime} \sin \theta^{\prime} d \theta^{\prime} \hat{z} \\
& =2 \pi k R^{3} \int \cos ^{2} \theta^{\prime} \sin \theta^{\prime} d \theta^{\prime} \hat{z} \\
& =2 \pi k R^{3} \int_{0}^{\pi} \cos ^{2} \theta^{\prime} \sin \theta^{\prime} d \theta^{\prime} \hat{z} \\
& =2 \pi k R^{3} \int_{-1}^{1} \cos ^{2} \theta^{\prime} d\left(\cos \theta^{\prime}\right) \hat{z} \\
& =2 \pi k R^{3}\left[\frac{\cos ^{3} \theta^{\prime}}{3}\right]_{-1}^{1} \hat{z} \\
& =2 \pi k R^{3}\left[\frac{1}{3}-\frac{-1}{3}\right] \hat{z} \\
& =\frac{4}{3} \pi k R^{3} \hat{z}
\end{aligned}
$$

## Part (b)

We can use the multipole expansion to find the approximate potential at points far from the sphere. The first term in this expansion is that due to a monopole.

$$
V_{m o n}=\frac{Q_{\text {total }}}{4 \pi \epsilon_{0} r}
$$

Use the charge density to determine $Q_{\text {total }}$.

$$
\begin{aligned}
Q_{\text {total }} & =\int \sigma d a \\
& =\int k \cos \theta^{\prime} r^{\prime 2} \sin \theta^{\prime} d \theta^{\prime} d \phi^{\prime} \\
& =2 \pi k R^{2} \int_{-1}^{1} \cos \theta^{\prime} d\left(\cos \theta^{\prime}\right) \\
& =2 \pi k R^{2}\left[\frac{\cos ^{2} \theta^{\prime}}{2}\right]_{-1}^{1} \\
& =2 \pi k R^{2}\left[\frac{1}{2}-\frac{1}{2}\right] \\
& =0
\end{aligned}
$$

Therefore the monopole term is zero. Next we calculate the dipole term that is given by eq. 3.102.

$$
\begin{aligned}
V_{d i p} & =\frac{\hat{r} \cdot \vec{p}}{4 \pi \epsilon_{0} r^{2}} \\
& =\frac{1}{4 \pi \epsilon_{0} r^{2}}\left(\frac{4}{3} \pi k R^{3} \hat{z} \cdot \hat{r}\right) \\
& =\frac{k R^{3}}{3 \epsilon_{0} r^{2}}[(\cos \theta \hat{r}-\sin \theta \hat{\theta} \cdot \hat{r}] \\
& =\frac{k R^{3}}{3 \epsilon_{0} r^{2}} \cos \theta
\end{aligned}
$$

The exact answer given in eq. 3.87 is,

$$
V(r, \theta)=\frac{k R^{3}}{3 \epsilon_{0} r^{2}} \cos \theta
$$

this is the same as our solution using only the first two terms of the multipole expansion. We may conclude that all of the higher multipole terms are zero.

## [2.] Problem 3.32 from Griffiths

Reference the distribution of three point charges as shown in Griffiths figure 3.38. Find the approximae electric field far from the origin. Solve this problem in spherical coordinates and use the two lowest orders in the multipole expansion.
There is a net charge in this configuration so we know that the monopole term is non-zero. Summing the charges we find $Q_{\text {total }}=-q$ and far away from these charges the electric field due to this monopole contribution will be,

$$
E_{\text {mono }} \approx \frac{-q}{4 \pi \epsilon_{0} r^{2}} \hat{r}
$$

Find the dipole moment to determine whether the dipole term for the electric field is nonzero.

$$
\begin{aligned}
\vec{p} & =\sum_{i}^{n} q_{i}{\overrightarrow{r^{\prime}}}_{i} \quad \text { Eq. } 3.100 \\
& =-q(-a \hat{y})+(-q)(a \hat{y})+q(a \hat{z}) \\
& =q a \hat{z} \\
& =q a(\cos \theta \hat{r}-\sin \theta \hat{\theta})
\end{aligned}
$$

This dipole moment is centered at the origin so we can use eq. 3.103 for the electric field of a dipole of this type. In this formulation the dipole moment is given as a magnitude (you will find that the magnitude of the dipole moment we found has no $\theta$ dependence, as should be expected). Equation 3.103 is exact for a pure dipole but only approximate for regions far away from a physical dipole.

$$
\begin{align*}
\vec{E}_{d i p} & \approx \frac{p}{4 \pi \epsilon_{0} r^{3}}(2 \cos \theta \hat{r}+\sin \theta \hat{\theta})  \tag{Eq. 3.103}\\
& \approx \frac{q a}{4 \pi \epsilon_{0} r^{3}}[2 \cos \theta \hat{r}+\sin \theta \hat{\theta}]
\end{align*}
$$

Both the monopole and dipole terms are non-zero so we have found the two lowest orders of the multipole expansion. Electric fields obey the superposition principle so the final solution for the approximate electric field far away from the point charges is,

$$
\vec{E} \approx \frac{q}{4 \pi \epsilon_{0}}\left[\left(\frac{2 a \cos \theta}{r^{3}}-\frac{1}{r^{2}}\right) \hat{r}+\frac{a \sin \theta}{r^{3}} \hat{\theta}\right]
$$

## [3.] Problem 3.33 from Griffiths

Show that the coordinate free form of the electric field due to a dipole may be written as,

$$
\vec{E}_{d i p}=\frac{1}{4 \pi \epsilon_{0} r^{3}}[3(\vec{p} \cdot \hat{r}) \hat{r}-\vec{p}] \quad \text { Eq. } 3.104
$$

## Method 1

Begin with a generic dipole moment. This is a simple vector and it can be written in terms of two coordinates, be they $x$ and $y$ (Cartesian) or r and $\theta$ (spherical). If we choose to write the dipole moment, $\vec{p}$ in terms of spherical coordinates then we can always put it in a form that has no $\phi$ dependence. Writing this vector in a general sense it would look like the following.

$$
\vec{p}=(\vec{p} \cdot \hat{r}) \hat{r}+(\vec{p} \cdot \hat{\theta}) \hat{\theta}
$$

Since the above way or writing the dipole moment is entirely general it must be an equivalent way of writing $\vec{p}$ for the dipole displayed in Griffiths, figure 3.36. This dipole moment
is directed entirely along the z-axis and may be written (using $\hat{z}=\cos \theta \hat{r}-\sin \theta \hat{\theta}$ ),

$$
\begin{aligned}
\vec{p} & =p \hat{z} \\
& =p \cos \theta \hat{r}-p \sin \theta \hat{\theta}
\end{aligned}
$$

Equating this with the general expression we see that $(\vec{p} \cdot \hat{r}) \hat{r}=p \cos \theta \hat{r}$ and we can go on to show,

$$
\begin{aligned}
3(\vec{p} \cdot \hat{r}) \hat{r}-\vec{p} & =3 p \cos \theta \hat{r}-p \cos \theta \hat{r}+p \sin \theta \hat{\theta} \\
& =2 p \cos \theta \hat{r}+p \sin \theta \hat{\theta}
\end{aligned}
$$

This is the term found in Eq. 3.103 and shows that the coordinate free form will return the correct answer for the example dipole given.

## Method 2

Take the general (coordinate free) expression for the potential due to a dipole and calculate the electric field.

$$
V_{d i p}=\frac{1}{4 \pi \epsilon_{0}} \frac{\vec{p} \cdot \hat{r}}{r^{2}} \quad \text { Eq. } 3.99
$$

The electric field is the negative gradient of the potential (see Griffiths page 78 for a reminder).

$$
\begin{aligned}
\vec{E}_{d i p} & =-\vec{\nabla} V \\
& =\frac{-1}{4 \pi \epsilon_{0}} \vec{\nabla}\left(\frac{\vec{p} \cdot \hat{r}}{r^{2}}\right) \\
& =\frac{-1}{4 \pi \epsilon_{0}} \vec{\nabla}\left(\vec{p} \cdot \frac{\hat{r}}{r^{2}}\right)
\end{aligned}
$$

Use this vector product rule (number 4 in the front cover of Griffiths),

$$
\vec{\nabla}(\vec{a} \cdot \vec{b})=(\vec{a} \cdot \vec{\nabla}) \vec{b}+(\vec{b} \cdot \vec{\nabla}) \vec{a}+\vec{a} \times(\vec{\nabla} \times \vec{b})+\vec{b} \times(\vec{\nabla} \times \vec{a})
$$

The gradient term in the electric field equation may be written,

$$
\vec{\nabla}\left(\vec{p} \cdot \frac{\hat{r}}{r^{2}}\right)=(\vec{p} \cdot \vec{\nabla}) \frac{\hat{r}}{r^{2}}+\left(\frac{\hat{r}}{r^{2}} \cdot \vec{\nabla}\right) \vec{p}+\vec{p} \times\left(\vec{\nabla} \times \frac{\hat{r}}{r^{2}}\right)+\frac{\hat{r}}{r^{2}} \times(\vec{\nabla} \times \vec{p})
$$

This expression can be simplified term by term. Begin with the last term on the right side. The dipole moment, $\vec{p}$ is a vector that is constant in space. Therefore, $\vec{\nabla} \times \vec{p}=0$ and this entire term is zero.

For the third term: We are still dealing with a coordinate free form, but the vector $\hat{r}$ is always equivalent to the $\vec{r}$ in spherical coordinates because it simply represents the position
of the object we care about. The curl of the $\hat{r}$ term is exactly like the curl of the electric field in our electrostatic cases, and we know that in electrostatics $\vec{\nabla} \times \vec{E}=0$.

$$
\begin{aligned}
\vec{\nabla} \times \vec{E}=0 & =\vec{\nabla} \times \frac{Q_{t o t}}{4 \pi \epsilon_{0} r^{2}} \hat{r} \\
& =\frac{Q_{t o t}}{4 \pi \epsilon_{0}} \vec{\nabla} \times \frac{\hat{r}}{r^{2}}
\end{aligned}
$$

So the third term is zero. Moving on to the second term we notice that it may be written as,

$$
\left(\frac{\hat{r}}{r^{2}} \cdot \vec{\nabla}\right) \vec{p}=\frac{1}{r^{2}} \nabla_{r} \vec{p}
$$

where $\nabla_{r}$ represents only the r term of the $\nabla$ operator. Again we may argue that $\vec{p}$ is constant in space and therfore any derivative the $\nabla_{r}$ term has will result in a value of zero when acted upon $\vec{p}$. This term is zero and the entire equation reduces to,

$$
\vec{\nabla}\left(\vec{p} \cdot \frac{\hat{r}}{r^{2}}\right)=(\vec{p} \cdot \vec{\nabla}) \frac{\hat{r}}{r^{2}}
$$

Once again considering a dipole directed along the z-axis (Griffiths Figure 3.36),

$$
\begin{aligned}
\vec{p} \cdot \vec{\nabla} & =(p \cos \theta \hat{r}-p \sin \theta \hat{\theta}) \cdot\left(\hat{r} \frac{\partial}{\partial r}+\hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta}+\hat{\phi} \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}\right) \\
& =p \cos \theta \frac{\partial}{\partial r}-p \sin \theta \frac{1}{r} \frac{\partial}{\partial \theta}
\end{aligned}
$$

Recall that $\frac{\partial \hat{r}}{\partial \theta}=\hat{\theta}$ to get,

$$
\begin{aligned}
(\vec{p} \cdot \vec{\nabla}) \frac{\hat{r}}{r^{2}} & =p \cos \theta\left(-\frac{2}{r^{3}}\right) \hat{r}-p \sin \theta\left(\frac{1}{r^{3}}\right) \hat{\theta} \\
& =\frac{1}{r^{3}}(-2 p \cos \theta \hat{r}-p \sin \theta \hat{\theta})
\end{aligned}
$$

Putting this into our expression for the electric field (there is another negative sign in that equation),

$$
\vec{E}=\frac{1}{4 \pi \epsilon_{0} r^{3}}(2 p \cos \theta \hat{r}+p \sin \theta \hat{\theta})
$$

and we found from Method 1 that $2 p \cos \theta \hat{r}+p \sin \theta \hat{\theta}=3(\vec{p} \cdot \hat{r}) \hat{r}-\vec{p}$. Thus we have again shown the equivalent form for the electric field due to a dipole.
Finally, there exist many (somewhat obscure) vector identities that could have been used to solve this problem. It is useful to have a reference for vector identities (Jackson, from the suggested texts for example). The one we could have used just above is,

$$
(\vec{a} \cdot \vec{\nabla}) \hat{r} f(r)=\frac{f(r)}{r}[\vec{a}-\hat{r}(\vec{a} \cdot \hat{r})]+\hat{r}(\vec{a} \cdot \hat{r}) \frac{\partial f(r)}{\partial r}
$$

Here $\vec{a}$ may be any vector and $f(r)$ must be continuous. We have $f(r)=\frac{1}{r^{2}}$ which is certainly not continuous at the origin $(r=0)$ but we are not concerned with the origin. We are trying to prove an expression for the dipole field at points far from the origin and in this case our $f(r)$ is acceptable. Using this identity we have,

$$
\begin{aligned}
(\vec{p} \cdot \vec{\nabla}) \frac{\hat{r}}{r^{2}} & =\frac{1}{r} \frac{1}{r^{2}}[\vec{p}-\hat{r}(\vec{p} \cdot \hat{r})]+\hat{r}(\vec{p} \cdot \hat{r})\left(\frac{-2}{r^{3}}\right) \\
& =\frac{1}{r^{3}}(\vec{p}-(\vec{p} \cdot \hat{r}) \hat{r})-\frac{2}{r^{3}}(\vec{p} \cdot \hat{r}) \hat{r} \\
& =\frac{1}{r^{3}}[-3(\vec{p} \cdot \hat{r}) \hat{r}+\vec{p}]
\end{aligned}
$$

Put this expression in our electric field equation (noting the minus sign in the electric field equation) and we see the coordinate free form is correct.

$$
\vec{E}_{d i p}=\frac{1}{4 \pi \epsilon_{0} r^{3}}[3(\vec{p} \cdot \hat{r}) \hat{r}-\vec{p}]
$$

## [4.] Problem 4.6 from Griffiths

A pure dipole is located a distance $z$ above an infinite grounded conducting plane (reference Griffiths figure 4.7). It makes an angle $\theta$ with the normal to the plane. Find the torque on the dipole. If it is free to rotate, then in what orientation will it come to rest?
The method of images may be applied to dipoles as well as point charges. Figure 2 illustrates the geometry of the image dipole $\overrightarrow{p^{\prime}}$ below the infinite plane. If you imagine the dipole as a positive charge on one end and a negative charge on the other end then you should be able to determine how to orient its image.
The torque on the $\vec{p}$ is given by,

$$
\begin{equation*}
\vec{N}=\vec{p} \times \vec{E} \tag{Eq. 4.4}
\end{equation*}
$$

where $\vec{E}$ is the electric field due to the infinite plane (or the image dipole) and not $\vec{p}$ itself. Find the electric field at $\vec{p}$ that is due to the image dipole, $\overrightarrow{p^{\prime}}$.

$$
\vec{E}=\frac{1}{4 \pi \epsilon_{0} r^{3}}\left[3\left(\overrightarrow{p^{\prime}} \cdot \hat{r}\right) \hat{r}-\overrightarrow{p^{\prime}}\right]
$$

Since $\hat{r}$ is directed toward $\vec{p}$ it is actually $\hat{z}$. The image dipole is given by (see figure 2) $\overrightarrow{p^{\prime}}=-p^{\prime} \sin \theta \hat{x}+p^{\prime} \cos \theta \hat{z}$. This fact coupled with $\overrightarrow{p^{\prime}} \cdot \hat{r}=p^{\prime} \cos \theta$ returns,

$$
\begin{aligned}
\vec{E} & =\frac{1}{4 \pi \epsilon_{0} r^{3}}\left[3 p^{\prime} \cos \theta \hat{z}-\left(-p^{\prime} \sin \theta \hat{x}+p^{\prime} \cos \theta \hat{z}\right)\right] \\
& =\frac{1}{4 \pi \epsilon_{0}(2 z)^{3}}\left[2 p^{\prime} \cos \theta \hat{z}+p^{\prime} \sin \theta \hat{x}\right]
\end{aligned}
$$

where in the last line I have used the fact that the distance between the dipoles is $r=2 z$.


Figure 2: Image dipole geometry.
The original dipole is described by, $\vec{p}=p \sin \theta \hat{x}+p \cos \theta \hat{z}$. The cross product is written in Cartesian coordinates as,

$$
\begin{aligned}
\vec{N} & =\vec{p} \times \vec{E} \\
& =\hat{x}\left(p_{y} E_{z}-p_{z} E_{y}\right)-\hat{y}\left(p_{x} E_{z}-p_{z} E_{x}\right)+\hat{z}\left(p_{x} E_{y}-p_{y} E_{x}\right) \\
& =-\hat{y}\left(p_{x} E_{z}-p_{z} E_{x}\right)
\end{aligned}
$$

since the y-component terms of both $\vec{p}$ and $\vec{E}$ are zero.
Plug in the values for the dipole and electric field components,

$$
\begin{aligned}
\vec{N} & =-\hat{y}\left(p \sin \theta \frac{2 p \cos \theta}{4 \pi \epsilon_{0}(2 z)^{3}}-p \cos \theta \frac{p \sin \theta}{4 \pi \epsilon_{0}(2 z)^{3}}\right) \\
& =\frac{-p^{2}}{32 \pi \epsilon_{0} z^{3}}[2 \sin \theta \cos \theta-\sin \theta \cos \theta] \hat{y} \\
& =\frac{-p^{2}}{32 \pi \epsilon_{0} z^{3}}[\sin \theta \cos \theta] \hat{y}
\end{aligned}
$$

Return to figure 2 again. From the right hand rule it is required that the $\hat{y}$ direction be into the page. Generally, if you point your right thumb in the direction of the torque, then your fingers point in the direction of the actual "twist". Do this to see that $\vec{p}$ is experiencing a torque out of the page and that results in a twisting counterclockwise as drawn in figure 2.

To determine the equilibrium position of $\vec{p}$ it is easier to examine the torque in terms of only one sinusoidal term. Since,

$$
\sin \theta \cos \theta=\frac{1}{2} \sin (2 \theta)
$$

we can write the torque as,

$$
\vec{N}=\frac{-p^{2}}{64 \pi \epsilon_{0} z^{3}} \sin (2 \theta)
$$

If $\theta=0$ then the torque on $\vec{p}$ is zero. A slight tilt of $\vec{p}$ either way results in a torque that seeks to rotate it back to the $\theta=0$ position. If $\theta=\pi$ then the torque on $\vec{p}$ is zero and any slight tilt will result in a torque that seeks to return it to $\theta=\pi$. The rest orientation of $\vec{p}$ is perpendicular to the infinite plane, either pointing up or down (i.e. along $\pm \hat{z}$ ).
Note: What happens at $\theta=\frac{\pi}{2}$ ? Here the torque on $\vec{p}$ is also zero yet we don't say that this is an equilibrium orientation. The torque on $\vec{p}$ is zero when $\vec{p}$ is parallel to the infinite plane, but any slight perturbation to this orientation results in a torque that pulls $\vec{p}$ to either the up or down orientation. Check the direction of the torque for angles a little above or below $\theta=\frac{\pi}{2}$ and you will see that this is not an equilibrium of the system. The same holds true for $\theta=\frac{3 \pi}{2}$.
[5.] See homework for problem statement
Part (a)
Begin by drawing the setup of the problem.


Figure 3: Diagram of dipoles for problem 5.
The script $O$ in figure 3 represents the origin of whatever coordinate system we are using. All other objects are drawn according to the way the problem is stated.
The energy of a dipole, $\vec{p}$, in and electric field, $\vec{E}$, is given by,

$$
U=-\vec{p} \cdot \vec{E} \quad \text { Eq. } 4.6
$$

The problem asks for the energy of $\vec{p}_{2}$ in the electric field of $\vec{p}_{1}$.

$$
\begin{aligned}
U=-\vec{p}_{2} \cdot \vec{E}_{1} & =-\vec{p}_{2} \cdot \frac{1}{4 \pi \epsilon_{0} r^{3}}\left[3\left(\vec{p}_{1} \cdot \hat{r}\right) \hat{r}-\vec{p}_{1}\right] \\
& \rightarrow \text { where } \quad r=R \quad \text { and } \quad \hat{r}=\hat{R} \\
& =\frac{1}{4 \pi \epsilon_{0} R^{3}} \vec{p}_{2} \cdot\left[\vec{p}_{1}-3\left(\vec{p}_{1} \cdot \hat{R}\right) \hat{R}\right] \\
& =\frac{1}{4 \pi \epsilon_{0} R^{3}}\left[\left(\vec{p}_{2} \cdot \vec{p}_{1}\right)-3\left(\vec{p}_{1} \cdot \hat{R}\right)\left(\vec{p}_{2} \cdot \hat{R}\right)\right]
\end{aligned}
$$

The $\vec{p}_{2}$ acts only the $\hat{R}$ in the second term because the quantity $\vec{p}_{1} \cdot \hat{R}$ is simply a scalar and does not matter for the dot product. Finally, the order of the vectors does not matter in dot (scalar) products so the first term above matches what we have been asked to show.

## Part (b)

The force on a dipole is given by,

$$
\begin{equation*}
\vec{F}=(\vec{p} \cdot \vec{\nabla}) \vec{E} \tag{Eq. 4.5}
\end{equation*}
$$

The force may also be written as the negative gradient of the potential energy, and in this problem the potential energy is the electrostatic energy of the dipole in an electric field.

$$
\vec{F}=-\vec{\nabla} U=\vec{\nabla}\left(\vec{p}_{2} \cdot \vec{E}_{1}\right)
$$

Since we have already solved for the energy of this system we can write the force on $\vec{p}_{2}$ due to $\vec{p}_{1}$ as,

$$
\vec{F}=\frac{1}{4 \pi \epsilon_{0}} \vec{\nabla}\left[\frac{1}{R^{3}}\left(3\left(\vec{p}_{1} \cdot \hat{R}\right)\left(\vec{p}_{2} \cdot \hat{R}\right)-\left(\vec{p}_{1} \cdot \vec{p}_{2}\right)\right]\right.
$$

Special Case 1)

$$
\vec{p}_{1} \| \vec{p}_{2} \quad \text { and } \quad \vec{p}_{1}, \vec{p}_{2} \perp \hat{R}
$$

This allows us to rewrite the dot products in our expression for the force. The dot product between a dipole moment and the $\hat{R}$ will be zero.

$$
\vec{F}=\frac{1}{4 \pi \epsilon_{0}} \vec{\nabla}\left[\frac{1}{R^{3}}\left(-p_{1} p_{2}\right)\right]
$$

The gradient term is with repsect only to the $\frac{1}{R^{3}}$ since the dipole values have no R dependence. You can verify this next result using the product rule twice.

$$
\vec{\nabla} \frac{1}{R^{3}}=\frac{-3}{R^{4}} \hat{R}
$$

Finally giving,

$$
\vec{F}=\frac{3 p_{1} p_{2}}{4 \pi \epsilon_{0} R^{4}} \hat{R}
$$

## Special Case 2)

$$
\vec{p}_{1}\left\|\vec{p}_{2}\right\| \hat{R}
$$

The dot products once again simplify.

$$
\begin{aligned}
\vec{F} & =\frac{1}{4 \pi \epsilon_{0}} \vec{\nabla}\left[\frac{2 p_{1} p_{2}}{R^{3}}\right] \\
& =\frac{1}{2 \pi \epsilon_{0}} \vec{\nabla}\left[\frac{p_{1} p_{2}}{R^{3}}\right] \\
& =\frac{-3 p_{1} p_{2}}{2 \pi \epsilon_{0} R^{4}} \hat{R}
\end{aligned}
$$

## [6.] Problem 4.9 from Griffiths

A dipole, $\vec{p}$ is located a distance, r , away from a point charge, q . The angle between $\vec{p}$ and $\vec{r}$ is $\theta$, where $\vec{r}$ points from q to $\vec{p}$.
(a) What is the force on $\vec{p}$ ? (b) What is the force on q ?

## Part (a)

Start by drawing a diagram of the problem. Figure 4 shows the locations of the dipole and point charge.
The force on the dipole due to the electric field generated by the point charge is,

$$
\vec{F}=(\vec{p} \cdot \vec{\nabla}) \vec{E} \quad \text { Eq. } 4.5
$$

Here the electric field due to the charge q is (shown in Cartesian cordinates because that will be easiest to show the math),

$$
\begin{aligned}
\vec{E} & =\frac{q}{4 \pi \epsilon_{0} r^{2}} \hat{r} \\
& =\frac{q}{4 \pi \epsilon_{0}} \frac{(x \hat{x}+y \hat{y}+z \hat{z})}{\left(x^{2}+y^{2}+z^{2}\right)^{3 / 2}}
\end{aligned}
$$

Noting that the dipole moment can always be written in terms of only two variables, we can say that $\vec{r}$ lies along the z-axis and $\vec{p}$ is in the $\mathbf{x z}$-plane. Then, $\vec{p}=p \sin \theta \hat{x}+p \cos \theta \hat{z}$.

$$
\begin{aligned}
\vec{p} \cdot \vec{\nabla} & =(p \sin \theta \hat{x}+p \cos \theta \hat{z}) \cdot\left(\hat{x} \frac{\partial}{\partial x}+\hat{y} \frac{\partial}{\partial y}+\hat{z} \frac{\partial}{\partial z}\right) \\
& =p \sin \theta \frac{\partial}{\partial x}+p \cos \theta \frac{\partial}{\partial z}
\end{aligned}
$$

You can review this representation of $\vec{\nabla}$ in Griffiths as Eq. 1.39.
Finally, the force is,

$$
\vec{F}=\frac{q}{4 \pi \epsilon_{0}}\left[p \sin \theta \frac{\partial}{\partial x}\left(\frac{x \hat{x}+y \hat{y}+z \hat{z}}{\left(x^{2}+y^{2}+z^{2}\right)^{3 / 2}}\right)+p \cos \theta \frac{\partial}{\partial z}\left(\frac{x \hat{x}+y \hat{y}+z \hat{z}}{\left(x^{2}+y^{2}+z^{2}\right)^{3 / 2}}\right)\right]
$$



Figure 4: Illustration of setup for problem 6.

I'll write out the math for the derivative with respect to $x$, and the $z$ derivative is found analogously.

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{x \hat{x}+y \hat{y}+z \hat{z}}{\left(x^{2}+y^{2}+z^{2}\right)^{3 / 2}}\right) & =\left(x^{2}+y^{2}+z^{2}\right)^{-3 / 2} \hat{x}+(x \hat{x}+y \hat{y}+z \hat{z})\left(-\frac{3}{2}\right)\left(x^{2}+y^{2}+z^{2}\right)^{-5 / 2}(2 x) \\
& =\frac{\left(x^{2}+y^{2}+z^{2}\right) \hat{x}-3 x(x \hat{x}+y \hat{y}+z \hat{z})}{\left(x^{2}+y^{2}+z^{2}\right)^{5 / 2}} \\
& =\frac{r^{2} \hat{x}-3 x \vec{r}}{r^{5}} \\
& =\frac{1}{r^{3}}\left[\hat{x}-\frac{3 x \hat{r}}{r}\right]
\end{aligned}
$$

The $z$ derivative will be just like this, but with the $z$ 's and $x^{\prime} s$ switched. The force becomes,

$$
\begin{aligned}
\vec{F} & =\frac{q}{4 \pi \epsilon_{0} r^{3}}\left[p \sin \theta\left(\hat{x}-\frac{3 x \hat{r}}{r}\right)+p \cos \theta\left(\hat{z}-\frac{3 z \hat{r}}{r}\right)\right] \\
& =\frac{q}{4 \pi \epsilon_{0} r^{3}}\left[p \sin \theta \hat{x}+p \cos \theta \hat{z}-3\left(\frac{p \sin \theta x+p \cos \theta z}{r}\right) \hat{r}\right] \\
& \rightarrow p \sin \theta x+p \cos \theta z=\vec{p} \cdot \vec{r} \\
& \rightarrow p \sin \theta \hat{x}+p \cos \theta \hat{z}=\vec{p} \\
& =\frac{q}{4 \pi \epsilon_{0} r^{3}}\left(\vec{p}-3\left(\frac{\vec{p} \cdot \vec{r}}{r}\right) \hat{r}\right) \\
& =\frac{q}{4 \pi \epsilon_{0} r^{3}}[\vec{p}-3(\vec{p} \cdot \hat{r}) \hat{r}]
\end{aligned}
$$

## Part (b)

Write the force on the point charge due to the dipole as $\vec{F}_{q}$ so that it does not get confused with the force we just found. The force on any point charge is given similarly,

$$
\vec{F}_{q}=q \vec{E}_{d i p}
$$

where $\vec{E}_{d i p}$ is the electric field due to only the dipole.
Recall once again from Eq. 3.104,

$$
\vec{E}_{d i p}=\frac{1}{4 \pi \epsilon_{0} r^{3}}[3(\vec{p} \cdot \hat{r}) \hat{r}-\vec{p}]
$$

The solution for the force on the point charge is simple in this case,

$$
\vec{F}_{d i p}=\frac{q}{4 \pi \epsilon_{0} r^{3}}[3(\vec{p} \cdot \hat{r}) \hat{r}-\vec{p}]
$$

Notice that $\vec{F}=-\vec{F}_{q}$ as it should.

