Massachusetts Institute of Technology
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Physics 8.07 Fall 2005

Problem Set 11 Solutions

Problem 1: Griffiths Problem 10.25 (p. 442)

The fields are

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{(1-\beta^2)}{(1-\beta^2\sin^2\theta)^{3/2}}\frac{\hat{r}}{r^2}\ , \quad \vec{B} = \frac{1}{c^2}\vec{v}\times\vec{E}\ ,$$

where $\beta = v/c$ and $\vec{r} = \vec{x} - \vec{v}t$ with $\vec{v} = v\vec{e}_x$. The Poynting vector is

$$\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0} = \left(\frac{q}{4\pi r^2}\right)^2 \frac{v(1-\beta^2)^2}{\epsilon_0(1-\beta^2\sin^2\theta)^3}[\hat{r}\times(\vec{e}_x\times\hat{r})]\ .$$

At $t = 0$, the power per unit area passing through a plane of constant x is

$$S_x(t=0) = \frac{\vec{E}\times\vec{B}}{\mu_0} = \left(\frac{q}{4\pi r^2}\right)^2 \frac{v(1-\beta^2)^2\sin^2\theta}{\epsilon_0(1-\beta^2\sin^2\theta)^3}\ .$$

To get the total power, we integrate $S_x dA$ where dA is the area element in the plane $x = a$. Let $s = r\sin\theta = a\tan\theta$ be the cylindrical radius measured from the x axis (instead of the usual z axis). Then $dA = 2\pi s\,ds = \pi a^2 d(\tan^2\theta)$. Let $u = \tan^2\theta$. The complete plane is spanned by $0 < u < \infty$. The distance and angles are related to u by

$$r^2 = a^2(1+\tan^2\theta) = a^2(1+u)\ , \quad \sin^2\theta = \frac{u}{1+u}\ .$$

Thus the power flowing through the surface at $t = 0$ is

$$P = \int S_x\,dA = \left(\frac{q}{4\pi a^2}\right)^2 \frac{\pi a^2 v(1-\beta^2)^2}{\epsilon_0}\int_0^\infty \frac{u\,du}{[1+(1-\beta^2)u]^3}\ .$$

Let $w = u(1-\beta^2)$; then

$$P = \frac{q^2 v}{16\pi\epsilon_0 a^2}\int_0^\infty \frac{w\,dw}{(1+w)^3} = \frac{q^2 v}{32\pi\epsilon_0 a^2}\ .$$

Problem 2: Griffiths 11.4 (p. 450)

For harmonic motion, $d^2\vec{p}/dt^2 = -\omega^2\vec{p}$, which is to be evaluated at $t_r = t - r/c$. For a dipole at $r = 0$, $\hat{r} = \vec{e}_r$, so

$$\vec{E} = -\frac{\mu_0\omega^2}{4\pi r}[\vec{e}_r \times (\vec{e}_r \times \vec{p})] \ , \quad \vec{B} = \frac{1}{c}\vec{e}_r \times \vec{E} \ .$$

In spherical coordinates, $\vec{p} = p_r\vec{e}_r + p_\theta\vec{e}_\theta + p_\phi\vec{e}_\phi$, and

$$\vec{e}_r \times (\vec{e}_r \times \vec{p}) = p_r\vec{e}_r - \vec{p} = -p_\theta\vec{e}_\theta - p_\phi\vec{e}_\phi \ , \quad \vec{e}_r \times [\vec{e}_r \times (\vec{e}_r \times \vec{p})] = p_\phi\vec{e}_\theta - p_\theta\vec{e}_\phi \ ,$$

giving

$$\vec{E} = \frac{\mu_0\omega^2}{4\pi r}(p_\theta\vec{e}_\theta + p_\phi\vec{e}_\phi) \ , \quad \vec{B} = \frac{\mu_0\omega^2}{4\pi c r}(-p_\phi\vec{e}_\theta + p_\theta\vec{e}_\phi) \ .$$

The spherical components are evaluated using

$$\begin{aligned}
\vec{e}_r &= \vec{e}_x \sin\theta\cos\phi + \vec{e}_y \sin\theta\sin\phi + \vec{e}_z\cos\theta \ , \\
\vec{e}_\theta &= \vec{e}_x \cos\theta\cos\phi + \vec{e}_y \cos\theta\sin\phi - \vec{e}_z\sin\theta \ , \\
\vec{e}_\phi &= -\vec{e}_x \sin\phi + \vec{e}_y\cos\phi \ .
\end{aligned}$$

With $p_x = p_0\cos\omega t_r$ and $p_y = p_0\sin\omega t_r$, we find

$$\begin{aligned}
p_\theta &= p_0\cos\theta(\cos\omega t_r\cos\phi + \sin\omega t_r\sin\phi) = p_0\cos\theta\cos(\omega t_r - \phi) \ , \\
p_\phi &= p_0(\sin\omega t_r\cos\phi - \cos\omega t_r\sin\phi) = p_0\sin(\omega t_r - \phi) \ .
\end{aligned}$$

The Poynting vector is now

$$\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0} = \frac{E^2}{\mu_0 c}\vec{e}_r = \frac{\mu_0 p_0^2\omega^4}{16\pi^2 c r^2}(\cos^2\theta\cos^2\alpha + \sin^2\alpha)\vec{e}_r \ , \quad \alpha \equiv \omega t_r - \phi \ .$$

Time-averaging gives $\langle\cos^2\alpha\rangle = \langle\sin^2\alpha\rangle = \frac{1}{2}$, so the intensity is

$$I = \langle S_r\rangle = \frac{\mu_0 p_0^2\omega^4}{32\pi^2 c r^2}(1 + \cos^2\theta) \ .$$

The intensity is maximal along the z-axis, i.e. perpendicular to the plane of the rotating dipole. This makes sense because for a single dipole the radiation is most intense in directions perpendicular to the dipole. The total power radiated is

$$P = \int I\, r^2 d\Omega = \frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\int(1+\cos^2\theta)\, d\Omega = \frac{\mu_0 p_0^2\omega^4}{6\pi c} \ ,$$

where we used $\int(1+\cos^2\theta)\, d\Omega = 4\pi\langle 1+\cos^2\theta\rangle = 16\pi/3$ (here the angle brackets denote an angular average, not a time average).

The power is twice that of a single dipole (Griffiths Eq. 11.22) because the x- and y-components are out of phase. To see this, let $\vec{p} = \vec{p}_1 + \vec{p}_2$ with $\vec{p}_1 = p_x\vec{e}_x$ and $p_2 = p_y\vec{e}_y$. Then squaring the electric field involves the combination

$$|\vec{e}_r \times (\vec{p}_1 + \vec{p}_2)|^2 = |\vec{e}_r\times\vec{p}_1|^2 + |\vec{e}_r\times\vec{p}_2|^2 + 2(\vec{e}_r\times\vec{p}_1)\cdot(\vec{e}_r\times\vec{p}_2) \ .$$

The time average of the last term is proportional to $\langle p_x p_y\rangle = p_0^2\langle\cos\omega t_r\sin\omega t_r\rangle = 0$ so the intensities of the two out-of-phase dipoles indeed add linearly.

Problem 3: Griffiths 11.5 (p. 454)

We start from Griffiths Eq. (11.33):

$$\vec{A} = \frac{\mu_0 m_0 \sin\theta}{4\pi r}\left(\frac{1}{r}\cos\omega t_r - \frac{\omega}{c}\sin\omega t_r\right)\vec{e}_\phi\ , \quad t_r \equiv t - \frac{r}{c}\ .$$

Using this, evaluate the fields:

$$\begin{aligned}
\vec{E} &= -\frac{\partial \vec{A}}{\partial t} = \frac{\mu_0 m_0 \omega^2 \sin\theta}{4\pi c r}\left(\cos\omega t_r + \frac{c}{\omega r}\sin\omega t_r\right)\vec{e}_\phi\ , \\
\vec{B} &= \vec{\nabla}\times\vec{A} = B_r\vec{e}_r + B_\theta\vec{e}_\theta\ , \\
& \quad B_r = \frac{\mu_0 m_0 \omega \cos\theta}{2\pi c r^2}\left(\frac{c}{\omega r}\cos\omega t_r - \sin\omega t_r\right)\ , \\
& \quad B_\theta = -\frac{\mu_0 m_0 \omega^2 \sin\theta}{4\pi c^2 r}\left[\left(1 - \frac{c^2}{\omega^2 r^2}\right)\cos\omega t_r + \frac{c}{\omega r}\sin\omega t_r\right]\ .
\end{aligned}$$

The electric field is the same as in Griffiths Problem 9.33 with $\omega t_r = \omega t - kr$, $k = \omega/c$, $A = \mu_0 m_0 \omega^2/4\pi c$. The Poynting vector is

$$\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0} = \frac{E_\phi}{\mu_0}(B_r\vec{e}_\theta - B_\theta\vec{e}_r)\ .$$

The needed time averages are

$$\left\langle\left(\cos\omega t_r + \frac{c}{\omega r}\sin\omega t_r\right)\left(\frac{c}{\omega r}\cos\omega t_r - \sin\omega t_r\right)\right\rangle = \frac{c}{\omega r}\frac{(1-1)}{2} = 0$$

and

$$\left\langle\left(\cos\omega t_r + \frac{c}{\omega r}\sin\omega t_r\right)\left[\left(1 - \frac{c^2}{\omega^2 r^2}\right)\cos\omega t_r + \frac{c}{\omega r}\sin\omega t_r\right]\right\rangle = \frac{1}{2}\left(1 - \frac{c^2}{\omega^2 r^2} + \frac{c^2}{\omega^2 r^2}\right) = \frac{1}{2}\ .$$

Thus,

$$\langle\vec{S}\rangle = I\vec{e}_r \quad \text{with} \quad I = -\frac{\langle E_\phi B_\theta\rangle}{\mu_0} = \frac{\mu_0}{2c}\left(\frac{m_0\omega^2\sin^2\theta}{4\pi c r}\right)^2\ ,$$

in complete agreement with Griffiths Eq. (11.39).

Problem 4: Griffiths 11.21 (p. 473)

a) The time-average flux per unit area striking the surface is

$$I = \left\langle\vec{S}_{\rm rad}\right\rangle\cdot\vec{e}_z = \frac{dP}{d\Omega}\frac{\cos\theta}{R^2+h^2}\ ,$$

where $\cos\theta = h/\sqrt{R^2+h^2}$ is the angle between $\hat{r}$ and $\vec{e}_z$. (Assuming that the displacement is small, the distance from the mass to the surface is $\sqrt{R^2+h^2}$, and $dP/d\Omega$ is the radial Poynting flux multiplied by the square of the distance.) Plugging in Eq. (11.74) in the limit $\beta \ll 1$, we get

$$I = \frac{\mu_0 q^2 \sin^2\theta \cos\theta}{16\pi^2 c(R^2+h^2)} \left\langle \ddot{z}^2 \right\rangle ,$$

where $\ddot{z}$ is the acceleration of the charge. The charge undergoes simple harmonic motion with $z(t) = d\cos\omega t$, so $\langle \ddot{z}^2 \rangle = \frac{1}{2}(d\omega^2)^2$. Putting it all together and using $\sin\theta = R/\sqrt{R^2+h^2}$, we get

$$I = \frac{\mu_0 (qd\omega^2)^2 R^2 h}{32\pi^2 c(R^2+h^2)^{5/2}} .$$

Maximizing this with respect to R gives $R = h\sqrt{2/3}$.

b) Using $R = h\tan\theta$, $dR = h\,d\theta/\cos^2\theta$, the total time-average power striking the floor is

$$P_{\text{floor}} = \int_0^\infty I\, 2\pi R\, dR = \frac{\mu_0 (qd\omega^2)^2}{16\pi c} \int_0^{\pi/2} \sin^3\theta\, d\theta = \frac{\mu_0 (qd\omega^2)^2}{24\pi c} .$$

This is exactly half the time-average power radiated, Griffiths Eq. (11.22) (with $p_0 = qd$). It's exactly what is to be expected, because half of the radiation goes down (to hit the floor) and half goes up.

c) Using $dE/dt = -P$, and $E = \frac{1}{2}m\omega^2 d^2$, we have

$$\frac{dE}{dt} = -2P_{\text{floor}} = -\frac{\mu_0 q^2 \omega^2}{6\pi m c} E ,$$

whose solution is

$$E = E_0 e^{-2t/\tau} \ , \quad \tau = \frac{12\pi m c}{\mu_0 q^2 \omega^2} .$$

The amplitude is $d = d_0 e^{-t/\tau}$, i.e. τ is the time for the amplitude to decrease by a factor e.

Problem 5: Griffiths 11.23 (p. 474)

a) The total power radiated by a magnetic dipole is

$$P = \frac{\mu_0}{6\pi c^3} \left\langle \left| \frac{d^2\vec{m}}{dt^2} \right|^2 \right\rangle$$

where the angle brackets denote a time average. For a magnetic dipole vector of fixed magnitude precessing with angular frequency ω around the z-axis,

$$\frac{d^2\vec{m}}{dt^2} = \vec{\omega} \times (\vec{\omega} \times \vec{m}\,) = -\omega^2\vec{m} + (\vec{\omega} \cdot \vec{m}\,)\vec{\omega} \ ,$$

where $\vec{\omega} = \omega\vec{e}_z$. Squaring this gives

$$\left|\frac{d^2\vec{m}}{dt^2}\right|^2 = M^2\omega^4(1 - \cos^2\psi) = M^2\omega^4 \sin^2\psi$$

where $M = |\vec{m}\,|$ and ψ is the angle between $\vec{m}$ and $\vec{\omega}$. The square is already independent of time, so the total power radiated by a precessing magnetic dipole is

$$P = \frac{\mu_0 M^2\omega^4}{6\pi c^3}\sin^2\psi \ .$$

b) A static magnetic dipole has field given by Griffiths Eq. (5.86). Evaluating this at the equator ($\theta = \pi/2$) gives

$$M = \frac{4\pi r^3}{\mu_0}B_{\mathrm{dip}} = \frac{4\pi(6.378 \times 10^6\,\mathrm{m})^3(0.5 \times 10^{-4}\,\mathrm{T})}{4\pi \times 10^{-7}\,\mathrm{T}\cdot\mathrm{m/A}} = 1.30 \times 10^{23}\,\mathrm{A}\cdot\mathrm{m}^2 \ .$$

The units of μ_0 were converted using Ampére's law to show that $1\,\mathrm{N/(A{\cdot}m)} = 1\,\mathrm{T}$.

c) The power emitted by magnetic dipole radiation is

$$\begin{aligned} P &= \frac{(4\pi \times 10^{-7}\,\mathrm{T}\cdot\mathrm{m/A})(1.30 \times 10^{23}\,\mathrm{A}\cdot\mathrm{m}^2)^2(2\pi\,\mathrm{s}^{-1}/86400)^4}{6\pi(2.998 \times 10^8\,\mathrm{m\,s}^{-1})^3}\sin^2\psi \\ &= 1.17 \times 10^{-3}\sin^2(11^\circ)\,\mathrm{w} = 4.3 \times 10^{-5}\,\mathrm{w} \ . \end{aligned}$$

d) The power emitted by a spinning neutron star is most easily obtained by scaling from the earth using

$$P \propto \left(\frac{B_{\mathrm{dip}}r^3}{T^2}\right)^2$$

where B_{dip} is the surface magnetic field at the equator, r is the radius, and T is the spin period. Putting in the numbers,

$$P_{\mathrm{pulsar}} = P_{\mathrm{earth}}\left(\frac{10^8\,\mathrm{T}}{0.5 \times 10^{-4}\,\mathrm{T}}\right)^2\left(\frac{10\,\mathrm{km}}{6378\,\mathrm{km}}\right)^6\left(\frac{86400\,\mathrm{s}}{10^{-3}\,\mathrm{s}}\right)^4 = 1.4 \times 10^{35}\,\mathrm{w} \ .$$

The book gives a larger number because it assumed $\psi = \pi/4$ instead of $\psi = 11^\circ$ for the neutron star, which increases the result by a factor $\sin^2(\pi/4)/\sin^2(11^\circ)$

to $P = 1.9 \times 10^{36}$ w. We do not know the misalignment angle between the spin and the dipole moment of neutron stars. For comparison, the luminosity of the sun is 4×10^{26} w – pulsars can be more than 10^9 times as luminous as the sun! However, this energy does not travel freely through the galaxy because the radiation frequency is far below the plasma frequency. The energy is absorbed by the plasma creating a relativistic expanding cloud like the Crab Nebula. This enormous luminosity does not persist; the energy comes from the neutron star's rotation which quickly decreases as the rotational energy is radiated away.