Welcome to the archival Web page for U.C. Berkeley's **Physics 110A**, Section 1, Fall 2001. Email to: (Prof.) Mark Strovink, strovink@lbl.gov . I have a research web page, a standardized U.C. Berkeley web page, and a statement of research interests.

Many of the problem set solutions for this course were handwritten initially by Graduate Student Instructor Ed Burns. They were refined, typeset, and additional solutions were composed by this course's GSI Daniel Larson.

Course documents:
Most documents linked here are in PDF format and are intended to be displayed by Adobe Acrobat [Reader], version 4 or later (Acrobat will do a better job if you **un**check "Use Greek Text Below:" on File-Preferences-General).

General Information including schedules and rooms.
Course Outline.

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Final Exam	

Mark Strovink

Professor
Particle Experiment

Mark Strovink, Ph.D. 1970 (Princeton). Joined UC Berkeley faculty in 1973 (Professor since 1980). Elected Fellow of the American Physical Society; served as program advisor for Fermilab (chair), SLAC (chair), Brookhaven, and the U.S. Department of Energy; served as D-Zero Physics Coordinator (1997 & 1998).

Research Interests

I am interested in experiments using elementary particles to test discrete symmetries, absolute predictions and other fundamental tenets of the Standard Model. Completed examples include early measurement of the parameters describing charge parity (*CP*) nonconservation in *K* meson decay; establishment of upper limits on the quark charge radius and early observation of the effects of gluon radiation in deep inelastic muon scattering; and establishment of stringent limits on right-handed charged currents both in muon decay and in proton-antiproton collisions, the latter via the search for production of right-handed *W* bosons in the D-Zero experiment at Fermilab.

After the discovery in 1995 by CDF and D-Zero of the top quark, we measured its mass with a combined 3% error, yielding (with other inputs) a stringent test of loop corrections to the Standard Model and an early hint that the Higgs boson is light. If a Higgs-like signal is seen, we will need to measure the top quark mass more than an order of magnitude better in order to determine whether that signal arises from the SM Higgs.

Current Projects

A continuing objective is to understand better how to measure the top quark mass. Top quarks are produced mostly in pairs; each decays primarily to $b + W$. The b's appear as jets of hadrons. Each W decays to a pair of jets or to a lepton and neutrino. For top mass measurement the most important channels are those in which either one or both of the W's decay into an electron or muon. For the single-lepton final states, we developed in 1994-96 and applied in 1997 a new technique that suppresses backgrounds (mostly from single W production) without biasing the apparent top mass spectra. For the dilepton final states, where backgrounds and systematic errors are lower but two final-state neutrinos are undetected rather than one, a likelihood *vs.* top mass must be calculated for each event. During 1993-96 we developed a new prescription for this calculation that averages over the (unmeasured) neutrino rapidities, and we used it in 1997 to measure the top mass to ~7% accuracy in this more sparsely populated channel. In both channels, further improvements to measurement technique as well as accumulation of larger samples will be necessary.

While studying data from the 1992-1996 CDF and D-Zero samples that contain both an electron and a muon, we became aware of three events that cannot easily be attributed either to top quark decay or to backgrounds. Generally this is because the transverse momenta of the leptons (electrons, muons, and neutrinos as inferred from transverse momentum imbalance) are unexpectedly large. We anticipate confirming data *e.g.* from the D-Zero run that began in 2001.

Transverse momentum imbalance is a broad signature for new physics. For example, in many supersymmetric models, *R*-parity conservation requires every superparticle to decay eventually to a lightest superparticle that, like the neutrino, can be observed only by measuring a transverse momentum imbalance. Reliable detection of this signature is one of the severest challenges for collider detectors. D-Zero's uniform and highly segmented uranium/liquid argon calorimeter yields the best performance achieved so far. Building on that, we have developed a new approach to analysis of transverse momentum imbalance that, for a given efficiency, yields up to five times fewer false positives.

Recently we have grappled with the long-standing problem of searching with statistical rigor for new physics in samples that should be describable by Standard Model processes – when the signatures for new physics are *not* strictly predefined. We have identified plausible methods for performing this type of analysis, and have exercised them on D-Zero data, but the methods involve sacrifices in sensitivity that we are still working to mitigate.

Selected Publications

S. Abachi *et al*. (D-Zero Collaboration), "Search for right-handed *W* bosons and heavy *W*′ in proton-antiproton collisions at $\sqrt{s}$ = 1.8 TeV," *Phys. Rev. Lett*. **76**, 3271 (1996).

S. Abachi *et al*. (D-Zero Collaboration), "Observation of the top quark," *Phys. Rev. Lett*. **74**, 2422 (1995).

B. Abbott *et al*. (D-Zero Collaboration), "Direct measurement of the top quark mass," *Phys. Rev. Lett*. **79**, 1197 (1997); *Phys. Rev.* **D 58**, 052001 (1998).

B. Abbott *et al*. (D-Zero Collaboration), "Measurement of the top quark mass using dilepton events," *Phys. Rev. Lett*. **80**, 2063 (1998); *Phys. Rev.* **D 60**, 052001 (1999).

V.M. Abazov *et al*. (D-Zero Collaboration), "A quasi-model-independent search for new high p_T physics at D-Zero," *Phys. Rev. Lett*. **86**, 3712 (2001); *Phys. Rev.* **D 62**, 092004 (2000); *Phys. Rev.* **D 64**, 012004 (2001).

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University of California, Berkeley
Physics 110A, Section 1, Fall 2001 (Strovink)

GENERAL INFORMATION (25 Sep 01)

Web site for this course: http://d0lbln.lbl.gov/110af01-web.htm .

Instructors: Prof. **Mark Strovink**, 437 LeConte; (LBL) 486-7087; (home, before 10) 486-8079; (UC) 642-9685. Email: strovink@lbl.gov . Web: http://d0lbln.lbl.gov/ . Office hours: M 3:15-4:15, 5:30-6:30.
Mr. **Daniel Larson**, 281 LeConte, (UC) 642-5647. Email: dtlarson@socrates.berkeley.edu . Office hours (in 281 LeConte): Th 11:30-12:30, F 2-3.

Lectures: MWF 10:10-11:00 in 329 LeConte, and Tu 5:10-6:30 in 329 LeConte. The Tu 5:10-6:30 slot will be used occasionally during the semester for the midterm exams; for reviews and special lectures; and for lectures that substitute for those which would normally be delivered later in the week. Lecture attendance is strongly encouraged, since the course content is not exactly the same as that of the text.

Discussion Sections: W 4:10-5 in 409 Davis, and Th 4:10-5 in 385 LeConte. Begin in second week. Taught by Mr. Larson. You are especially encouraged to attend discussion section regularly. There you will learn techniques of problem solving, with particular application to the assigned exercises.

Texts:
• Griffiths, **Introduction to Electrodyamics** (3rd ed., Prentice-Hall, 1999, required). Get the fourth (or later) printing, which has fewer typos. I feel that this text is well written and pedagogically effective, though its scope is modest and its problems are sometimes not very physical.
• If you are planning to attend physics graduate school, it would be smart now to purchase Jackson, **Classical Electrodynamics** (3rd ed., Wiley). Optionally, it can be useful in this course.

Problem Sets: A required and most important part of the course. Eleven problem sets are assigned and graded. Problem sets are due on Fridays at 5 PM, beginning in week 2. *Exceptions*: no problem set is due in the week preceding each midterm exam; the problem set that normally would be due on Friday of the week of the second midterm exam instead is due four days later, on Tuesday of Thanksgiving week (no other problem set is due on Thanksgiving week). Deposit problem sets in the box labeled "110A Section 1 (Strovink)" in the second floor breezeway between LeConte and Birge Halls. You are encouraged to attempt all of the problems. Students who do not do so find it almost impossible to learn the material and to succeed on the examinations. Late papers will not be graded. Your lowest problem set score will be dropped, in lieu of due date extensions for any reason. You are encouraged to discuss problems with others in the course, but you must write up your homework by yourself. (It is straightforward to identify solutions that are written collectively; our policy is to divide the score among the collectivists.)

Exams: There will be two 80-minute midterm examinations and one 3-hour final examination. Before confirming your enrollment in this class, please check that its final Exam Group 1 does not conflict with the Exam Group for any other class in which you are enrolled. Please verify now that you will be available for the midterm examinations on Tu 16 Oct (in 4 LeConte) and Tu 13 Nov (in 50 Birge), both at 5:10-6:30 PM; and for the final examination on W 12 Dec, 8-11 AM. Except for unforeseeable emergencies, it will not be possible for the midterm or final exams to be rescheduled. Passing 110A requires passing the final exam.

Grading: 25% problem sets, 35% midterms, 40% final exam. Departmental regulations call for an A:B:C distribution in the ratio 2:3:2, with approximately 10-15% D's or F's. However, the fraction of D's or F's depends on you; no minimum number need be given.

Week No.	Week of...	Lecture reference (Griffiths)	Topic	Problem Set No.	Due 5 PM on...
1	27-Aug	1.1.5, 1.3.2-1.3.6	Vector and tensor transformations, fundamental theorems		
		1.4-1.6	Curvilinear coordinates, Dirac delta function, theory of vector fields		
		2.1-2.2.3	Electrostatic fields, Gauss's law		
2	3-Sep		LABOR DAY		
		2.2.4-2.3	Electrostatic potential and boundary conditions		
		2.4, 2.5.1	Electrostatic work and energy, conductors	1	7-Sep
3	10-Sep	3.1.1-3.1.4	Laplace's and Poisson's equation, simple and relaxation solutions		
		3.1.5, 3.2.1-3.2.2	Uniqueness of solution, method of images		
		3.3.1	Separation of variables in Cartesian coordinates	2	14-Sep
4	17-Sep	3.4.2, 3.4.4	Ideal electric dipole and its field		
		4.1-4.2.1	Forces and torques on electric dipoles; polarization		
		4.3-4.4.1	Gauss's law in dielectrics, **D**, linear dielectrics	3	21-Sep
5	24-Sep	4.4.3-4.4.4	Energy in dielectrics, forces on dielectrics		
		5.1.1-5.1.2	Lorentz force law, particle trajectories in static fields		
		5.1.3	Current, forces on wires, current densities; charge conservation	4	28-Sep
6	1-Oct	5.2, 5.3.1-5.3.2	Biot-Savart law, divergence of **B**		
		5.3.2-5.3.4	Ampere's law and applications, static Maxwell equations		
		5.4.1-5.4.2	Vector potential, magnetostatic boundary conditions	5	5-Oct
7	8-Oct	5.4.3	Ideal magnetic dipole and its field		
		6.1.1-6.1.2, 6.1.4	Forces and torques on magnetic dipoles; magnetization		
		6.3, 6.4.1	Ampere's law in magnetic materials, **H**, linear magnetic media		
8	15-Oct	---	TBA		
	(16-Oct)	---	MIDTERM 1 (covers PS 1-5), in 4 LeConte		
		6.4.2	Ferromagnetism	6	19-Oct
9	22-Oct	7.1	Ohm's law, EMF		
		7.2.1-7.2.2	Faraday's law		
		7.2.3-7.2.4	Energy in magnetic fields, inductance	7	26-Oct
10	29-Oct	7.3.1-7.3.3	Maxwell's equations in free space		
		7.3.5-7.3.6	Maxwell's equations in matter, boundary conditions		
		10.1	Maxwell's equations for potentials; gauge transformations	8	2-Nov
11	5-Nov	8.1.1-8.1.2	Continuity equation, Poynting's theorem		
		9.1.1-9.1.2	Wave equation in one dimension, general solution, sinusoidal waves		
		9.2	EM waves in vacuum, energy and momentum		
12	12-Nov	---	VETERANS DAY		
	(13-Nov)	---	MIDTERM 2 (covers PS 1-8), in 50 Birge		
		9.3.1-9.3.2	EM waves in a linear insulator, reflection at normal incidence		
14	19-Nov	11.1.1-11.1.2	EM fields of an oscillating electric dipole		
		11.1.1-11.1.2	Electric dipole radiation and power	9	20-Nov
	(22-Nov)	---	THANKSGIVING		
13	26-Nov	9.1.4	Polarization and angular momentum of EM waves; how to control		
		9.4.1-9.4.2	EM waves in a conductor, reflection at normal incidence		
		9.5.1, 9.5.3	EM waves in a coaxial cable	10	30-Nov
15	3-Dec	---	Interference and coherence of >1 dipole radiator		
		---	Radiation pattern from >1 dipole and connection to diffraction (Babinet)		
		---	TBA	11	7-Dec
16	10-Dec	---			
	(12-Dec)	---	Final exams begin		
	(12-Dec)	8-11 AM	110A FINAL EXAM (Group 1) (covers PS 1-12)		

University of California, Berkeley
Physics 110A Fall 2001 Section 1 (*Strovink*)

Problem Set 1

1. Griffiths 1.14

2. Griffiths 1.16

3. Griffiths 1.21

4. Griffiths 1.33

5. Griffiths 1.38

6. Griffiths 1.46

7. Griffiths 2.6

8. Griffiths 2.16

University of California, Berkeley
Physics 110A Fall 2001 Section 1 (*Strovink*)

Solution Set 1

1. **Griffiths 1.14** Under a rotation, the coordinates y and z transform into $\bar{y} = y\cos\phi + z\sin\phi$ and $\bar{z} = -y\sin\phi + z\cos\phi$, so we can invert these equations to find $y = \bar{y}\cos\phi - \bar{z}\sin\phi$ and $z = \bar{y}\sin\phi + \bar{z}\cos\phi$. Using the chain rule:

$$\begin{aligned}\frac{\partial f}{\partial \bar{y}} &= \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{y}} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \bar{y}} = \frac{\partial f}{\partial y}\cos\phi + \frac{\partial f}{\partial z}\sin\phi \\ \frac{\partial f}{\partial \bar{z}} &= \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \bar{z}} = \frac{\partial f}{\partial y}(-\sin\phi) + \frac{\partial f}{\partial z}\cos\phi.\end{aligned}$$

Thus
$$\begin{pmatrix} (\overline{\boldsymbol{\nabla} f})_y \\ (\overline{\boldsymbol{\nabla} f})_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} (\boldsymbol{\nabla} f)_y \\ (\boldsymbol{\nabla} f)_z \end{pmatrix} \Rightarrow (\overline{\boldsymbol{\nabla} f})_i = \mathcal{R}_{ij}(\boldsymbol{\nabla} f)_j.$$

So $\boldsymbol{\nabla} f$ transforms like a vector.

2. **Griffiths 1.16** The sketch of this vector field appears in Figure 1.44 of the text. (In fact, a whole discussion appears in Section 1.5.1.) We want to calculate the divergence of the vector field $\mathbf{v} = \hat{\mathbf{r}}/r^2 = (x,y,z)/r^3$. If you calculate the divergence in cartesian coordinates, the formula for the divergence is simple, $\boldsymbol{\nabla}\cdot\mathbf{v} = (\partial v_x/\partial x) + (\partial v_y/\partial y) + (\partial v_z/\partial z)$, however, the derivatives can get a little tiresome. Instead, I'll calculate the divergence in spherical coordinates. A general vector field in spherical coordinates looks like $\mathbf{v} = v_r\,\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}} + v_\phi\hat{\boldsymbol{\phi}}$ and the formula for the divergence is a little complicated (see the inside front cover of Griffiths):

$$\boldsymbol{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

However, in our case, $v_\theta = v_\phi - 0$, so the calculation isn't so bad. In fact, since $v_r = 1/r^2$, $\partial(r^2 v_r)/\partial r = \partial(1)/\partial r = 0$. Thus $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$! This is surprising, because the vector field certainly looks like it is diverging away from the origin. The explanation is that the divergence is zero everywhere *except* at the origin; at the origin the above calculation fails because the vector field is undefined there.

3. **Griffiths 1.21**

(a) $(\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} = (A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z})\mathbf{B}$
$= (A_x\frac{\partial B_x}{\partial x} + A_y\frac{\partial B_x}{\partial y} + A_z\frac{\partial B_x}{\partial z})\hat{\mathbf{x}} + (A_x\frac{\partial B_y}{\partial x} + A_y\frac{\partial B_y}{\partial y} + A_z\frac{\partial B_y}{\partial z})\hat{\mathbf{y}} + (A_x\frac{\partial B_z}{\partial x} + A_y\frac{\partial B_z}{\partial y} + A_z\frac{\partial B_z}{\partial z})\hat{\mathbf{z}}$.
This can be expressed more succintly using index notation and the summation convention: $[(\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B}]_i = A_j\partial_j B_i$.

(b) As in an earlier problem, the computation is a little easier in spherical coordinates; however, this time I'll use cartesian coordinates for variety. For the x component

$$\begin{aligned}[(\hat{\mathbf{r}}\cdot\boldsymbol{\nabla})\hat{\mathbf{r}}]_x &= \frac{1}{\sqrt{x^2+y^2+z^2}}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\frac{x}{\sqrt{x^2+y^2+z^2}} \\ &= \frac{1}{r}\left\{x\left[\frac{1}{r} + x\left(-\frac{1}{2}\right)\frac{1}{r^3}2x\right] + yx\left[-\frac{1}{2}\frac{1}{r^3}2y\right] + zx\left[-\frac{1}{2}\frac{1}{r^3}2z\right]\right\} \\ &= \frac{1}{r}\left\{\frac{x}{r} - \frac{x}{r^3}\left(x^2+y^2+z^2\right)\right\} = \frac{1}{r}\left(\frac{x}{r} - \frac{x}{r}\right) = 0.\end{aligned}$$

The other two components are the same, just swapping x for y or z. Thus all three components vanish, so $[(\hat{\mathbf{r}}\cdot\boldsymbol{\nabla})\hat{\mathbf{r}}] = 0$.

(c) Now take $\mathbf{v}_a = x^2\,\hat{\mathbf{x}} + 3xz^2\,\hat{\mathbf{y}} - 2xz\,\hat{\mathbf{z}}$ and $\mathbf{v}_b = xy\,\hat{\mathbf{x}} + 2yz\,\hat{\mathbf{y}} + 3zx\,\hat{\mathbf{z}}$.

$$\begin{aligned}(\mathbf{v}_a\cdot\boldsymbol{\nabla})\mathbf{v}_b &= \left(x^2\frac{\partial}{\partial x} + 3xz^2\frac{\partial}{\partial y} - 2xz\frac{\partial}{\partial z}\right)(xy\,\hat{\mathbf{x}} + 2yz\,\hat{\mathbf{y}} + 3xz\,\hat{\mathbf{z}} \\ &= x^2(y\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}} + 3z\,\hat{\mathbf{z}}) + 3xz^2(x\,\hat{\mathbf{x}} + 2z\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}) - 2xz(0\,\hat{\mathbf{x}} + 2y\,\hat{\mathbf{y}} + 3x\,\hat{\mathbf{z}}) \\ &= x^2(y + 3z^2)\,\hat{\mathbf{x}} + 2xz(3z^2 - 2y)\,\hat{\mathbf{y}} - 3x^2z\,\hat{\mathbf{z}}\end{aligned}$$

4. **Griffiths 1.33**. We want to test Stokes' Theorem for the function $\mathbf{v} = (xy)\,\hat{\mathbf{x}} + (2yz)\,\hat{\mathbf{y}} + (3zx)\,\hat{\mathbf{z}}$ over the triangular region shown in the figure. First let's calculate the integral of the curl over the triangle's area. $\boldsymbol{\nabla}\times\mathbf{v} = \hat{\mathbf{x}}(0-2y) + \hat{\mathbf{y}}(0-3z) + \hat{\mathbf{z}}(0-x) = -2y\,\hat{\mathbf{x}} - 3z\,\hat{\mathbf{y}} - x\,\hat{\mathbf{z}}$. Since the path around the outside is going counterclockwise, the convention is that the area element $d\mathbf{a} = dy\,dz\,\hat{\mathbf{x}}$. Thuse $(\boldsymbol{\nabla}\times\mathbf{v})\cdot d\mathbf{a} = -2y\,dy\,dz$.

$$\int(\boldsymbol{\nabla}\times\mathbf{v})\cdot d\mathbf{a} = \int_0^2\left(\int_0^{2-z} -2y\,dy\right)dz = \int_0^2 -(2-z)^2\,dz = \frac{1}{3}(2-z)^3\Big|_0^2 = -\frac{8}{3}.$$

Now we calculate the path integral around the boundary of the triangle. $\mathbf{v}\cdot d\mathbf{l} = (xy)\,dx + (2yz)\,dy + (3zx)\,dz$. There are three segments. Take the segment along the y-axis first. Here y goes from 0 to 2, but dx and dz are zero, and $x = y = 0$. Thus $\mathbf{v}\cdot d\mathbf{l} = 0$ so its integral is zero also. Let the second segment be the slanted side. Again, $x = dx = 0$, while $z = 2 - y$ and $dz = -dy$. In traveling up and left y goes from 2 to 0. So $\int\mathbf{v}\cdot d\mathbf{l} = \int_2^0 2yz\,dy = -\int_0^2 2y(2-y)\,dy = -(2y^2 - \frac{2}{3}y^3)\big|_0^2 = -\frac{8}{3}$. And the final segment is coming down the z-axis, where $dx = dy = 0$ and $x = y = 0$ so $\mathbf{v}\cdot d\mathbf{l} = 0$ and thus there is no contribution to the integral. So the contributions from all three segments give $\oint\mathbf{v}\cdot d\mathbf{l} = -\frac{8}{3}$. Thus in this case we have demonstrated that Stokes' theorem holds, namely $\int_S(\boldsymbol{\nabla}\times\mathbf{v})\cdot d\mathbf{a} = \oint_P\mathbf{v}\cdot d\mathbf{l}$.

5. **Griffiths 1.38** The divergence theorem tells us that the integral of the divergence over the volume equals the integral of the vector field over the surface. So we need to compute two integrals in each case.

   (a) $\boldsymbol{\nabla}\cdot(r^2\,\hat{\mathbf{r}}) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^4) = 4r$, where I used equation (1.71). Now, $\int(\boldsymbol{\nabla}\cdot\mathbf{v}_1)\,d\tau = \int(4r)r^2\,dr\,\sin\theta\,d\theta\,d\phi = \left(\int_0^R 4r^3\,dr\right)\left(\int d\Omega\right) = r^4\big|_0^R(4\pi) = 4\pi R^4$. On the other hand, on the surface of the sphere, $r = R$, and so $\int\mathbf{v}_1\cdot d\mathbf{a} = \int(R^2\,\hat{\mathbf{r}})\cdot(R^2\sin\theta\,d\theta, d\phi\,\hat{\mathbf{r}}) = R^4\int d\Omega = 4\pi R^4$. The two integrals agree.

   (b) $\boldsymbol{\nabla}\cdot(1/r^2)\,\hat{\mathbf{r}} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{1}{r^2}) = 0$, so $\int(\boldsymbol{\nabla}\cdot\mathbf{v}_2)\,d\tau = 0$. On the other hand, on the surface of a sphere radius R, $\int\mathbf{v}_2\cdot d\mathbf{a} = \int(\frac{1}{R^2}\,\hat{\mathbf{r}})\cdot(R^2 d\Omega\,\hat{\mathbf{r}}) = \int d\Omega = 4\pi$. The two integrals *don't* agree! The reason is that $\boldsymbol{\nabla}\cdot\mathbf{v}_2 = 0$ except at the origin, where it becomes infinite. Thus our calculation of $\int(\boldsymbol{\nabla}\cdot\mathbf{v}_2)\,d\tau$ is incorrect. We'll learn how to fix this using the Dirac delta-function. The correct answer is 4π, whic is what we got the using the surface integral because that method avoids the problem at the origin.

6. **Griffiths 1.46**

   (a) Since the charge is only at the specific point $\mathbf{r}'$ we will need to use a delta-function. $\rho(\mathbf{r}) = q\delta^3(\mathbf{r}-\mathbf{r}')$. The volume integral is then $\int\rho(\mathbf{r})\,d\tau = q\int\delta^3(\mathbf{r}-\mathbf{r}')\,d\tau = q$ as it should be.

   (b) The electric dipole is just two different point charges at different places. $\rho(\mathbf{r}) = q\delta^3(\mathbf{r}-\mathbf{a}) - q\delta^3(\mathbf{r})$.

   (c) There is no charge anywhere except where $r = R$, so $\rho(\mathbf{r}) = A\delta(r-R)$. We need to integrate over all space to find A. $Q = \int\rho\,d\tau = \int A\delta(r-R)4\pi r^2\,dr = A4\pi R^2$. So $A = \frac{Q}{4\pi R^2}$. Thus $\rho(r) = \frac{Q}{4\pi R^2}\delta(r-R)$.

7. **Griffiths 2.6** The first thing to notice is that any horizontal components of the electric field will cancel because of the symmetry of the disk. So the resulting electric field will be in the $\hat{\mathbf{z}}$ direction. Then we just need to add up the contributions of the z-components due to every point on the disk. For a generic point on the disk located at a distance r from the center, the distance to the point P is $\sqrt{r^2+z^2}$. Thus the z-component of the E field at P due to that point is $\frac{1}{4\pi\epsilon_0}\frac{dq}{r^2+z^2}\cos\theta$ where θ is the angle between the line connecting P with the center of the disk and the line connecting P with the generic element of charge on the disk (see Fig. 1). So we

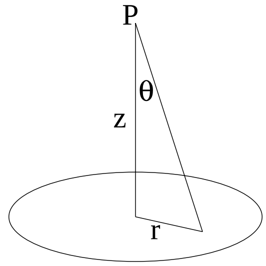

Figure 1: Problem 7.

see $\cos\theta = \frac{z}{\sqrt{r^2+z^2}}$. To get the total E-field at P we need to add up these contributions for every little element of charge on the disk. The amount of charge in a small area element is $dq = \sigma r dr d\phi$. Thus

$$\begin{aligned} E &= \int_0^R \int_0^{2\pi} \frac{1}{4\pi\epsilon_0} \frac{z\sigma r dr\, d\phi}{(r^2+z^2)^{3/2}} \hat{\mathbf{z}} = \frac{2\pi z\sigma}{4\pi\epsilon_0} \int_0^R \frac{r dr}{(r^2+z^2)^{3/2}} \hat{\mathbf{z}} \\ &= \frac{2\pi z\sigma}{4\pi\epsilon_0} \left[-(r^2+z^2)^{-1/2} \right]_0^R \hat{\mathbf{z}} = \frac{z\sigma}{2\epsilon_0} \left[\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}} \right] \hat{\mathbf{z}}. \end{aligned}$$

Finally, it is always good to check the limiting cases of your results. For $R \to \infty$ the second term vanishes and we're left with $\sigma/2\epsilon_0$ which is the electic field due to an infinite sheet of charge. For $z \gg R$ at first glance the two terms in brackets cancel, so we'd get zero, which is true, but not so informative. To get a more precise estimate of how the field falls off for large z we need to expand the square root.

$$\frac{1}{\sqrt{R^2+z^2}} = \left[z^2 \left(1 + \frac{R^2}{z^2} \right) \right]^{-1/2} = \frac{1}{z} \left(1 - \frac{R^2}{2z^2} + \mathcal{O}\left(\frac{R^4}{z^4} \right) \right)$$

Since $z \gg R$, we can ignore the R^4/z^4 piece. Plugging the expansion back into the formula for the electric field, we find $E \to \frac{\sigma\pi R^2}{4\pi\epsilon_0 z^2}$ which is the field a distance z away from a point charge with $Q = \sigma\pi R^2$.

8. **Griffiths 2.16** We can use Gauss's law with cylindrical surfaces to determine the electric field in each region. Following the notation in Griffiths, $\hat{\mathbf{s}}$ is the radial unit vector in cylindrical coordinates.

   (i) For $s < a$, we imagine a small cylindrical surface of radius s and length ℓ inside the inner cylinder. We know by symmetry that the resulting electric field must point radially outward and is thus perpendicular to the curved sides of our Gaussian cylinder. Thus $\oint \mathbf{E} \cdot d\mathbf{a} = 2\pi s\ell E(s)$. Gauss's law tells us this is equal to $\frac{1}{\epsilon_0} Q_{\text{enc}} = \frac{1}{\epsilon_0}\pi s^2 \ell\rho$. Equating these two expressions and solving for the electric field we find $\mathbf{E}(s) = \frac{s\rho}{2\epsilon_0}\hat{\mathbf{s}}$.

   (ii) Now we imagine our gaussian surface to be between the cylinders. The flux of electric field leaving the cylinder is the same as above, namely $\oint \mathbf{E} \cdot d\mathbf{a} = 2\pi s\ell E(s)$, while the other half of Gauss's law gives $\frac{1}{\epsilon_0} Q_{\text{enc}} = \frac{1}{\epsilon_0}\pi a^2 \ell\rho$. Thus for $a < s < b$, $\mathbf{E}(s) = \frac{a^2\rho}{2\epsilon_0 s}\hat{\mathbf{s}}$.

   (iii) $\oint \mathbf{E} \cdot d\mathbf{a} = 2\pi s\ell E(s) = \frac{1}{\epsilon_0} Q_{\text{enc}} = 0$ because we are told that the whole wire is neutral, so the enclosed charge on the central cylinder in cancelled by the enclosed charge on the outer cylindrical shell. Thus for $s > a$, $\mathbf{E} = 0$.

   The plot of the electric field as a function of s starts at zero, increases linearly until $s = a$, then it decreases along a segment of a hyperbola until $s = b$, and which point it abruptly drops to zero for all $s > a$.

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Problem Set 2

1. Griffiths 2.18

2. Griffiths 2.20

3. Griffiths 2.25 (c) only

4. Griffiths 2.32 (a) and (b) only

5. Griffiths 2.36 (a), (b), and (c) only

6. Griffiths 2.39

7. Griffiths 2.50

8. According to the Proca equations (a relativistically invariant linear generalization of Maxwell's equations accommodating the possibility of a finite rest mass m_0 for the photon), Gauss's law is modified to become

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} - \frac{\phi}{\bar{\lambda}^2} \, ,$$

where ϕ is the electrostatic potential and

$$\bar{\lambda} \equiv \frac{\hbar}{m_0 c}$$

is the reduced (by 2π) Compton wavelength of the photon.

Following Williams, Faller, and Hill, *Phys. Rev. Lett.* **26**, 721 (1971), consider two concentric spherical perfectly conducting shells of radii R_1 and R_2, respectively, with $R_2 > R_1$. Imagine that the inner sphere is isolated and that the outer shell is driven by an RF oscillator so that it has a potential (relative to ∞)

$$\phi_2(t) = V_0 \cos \omega t \, .$$

In the modified form of Gauss's law, make the following approximation for the value of ϕ which appears in the last term; this is a valid approach because the factor $\bar{\lambda}^{-2}$ multiplying it is very small. The approximation is to set $\phi = \phi_2$ everywhere within the outer sphere. Construct a Gaussian surface consisting of a third sphere at radius r, where $R_1 < r < R_2$. Consider the volume integral of

$$\nabla \cdot \mathbf{E} - \frac{\rho}{\epsilon_0} + \frac{\phi}{\bar{\lambda}^2}$$

within that surface. Using the divergence theorem, convert it to a surface integral over the Gaussian surface. Evaluate the integral to obtain the (radial) electric field at r. Your result should contain a term proportional to the charge q on the inner sphere, and another small term proportional to m_0^2. Integrate this electric field from R_1 to R_2 to solve for the potential difference v that would be measured between the inner and outer spheres.

Assuming that $q = 0$, $R_1 = 0.5$ m, and $R_2 = 1.5$ m, and that V_0 is 10 kV, find the voltage v between the inner and outer spheres that would be observed if the photon had a rest mass $m_0 = 10^{-15}$ eV$/c^2$.

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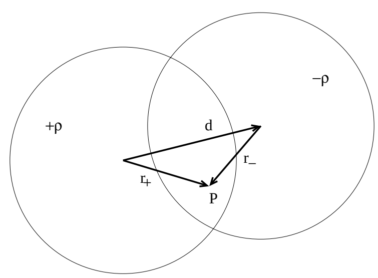
Solution Set 2

1. **Griffiths 2.18** First we need the electric field inside a uniformly charged sphere. Imagining a spherical Gaussian surface of radius r inside the charged sphere, symmetry tells us that the electric field must be pointing radially outward and have the same magnitude over the whole surface. So $\int \mathbf{E}(r)\cdot d\mathbf{a} = 4\pi r^2 E(r) = Q_{\text{enc}}/\epsilon_0 = \rho 4\pi r^3/3$. Thus $\mathbf{E}(r) = \rho r/(3\epsilon_0)\,\hat{\mathbf{r}} = \rho r/(3\epsilon_0)\,\mathbf{r}$ where $\mathbf{r}$ is the unit vector from the center of the charged sphere to the point in question.

   In this problem we have two charged spheres with the vector $\mathbf{d}$ pointing from the center of the positive sphere to the center of the negative sphere. For a point P in the region of overlap, there will be a contribution to the E-field from both spheres. If we let $\mathbf{r}_+$ be the vector from the center of the positive sphere to P and $\mathbf{r}_-$ be the vector from the center of the negative sphere to P, then the total E-field at P is

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = \frac{\rho}{3\epsilon_0}\mathbf{r}_+ + \frac{-\rho}{3\epsilon_0}\mathbf{r}_- = \frac{\rho}{3\epsilon_0}(\mathbf{r}_+ - \mathbf{r}_-).$$

   But from the figure, we see that $\mathbf{r}_+ - \mathbf{r}_- = \mathbf{d}$. So the total electric field in the overlap region is $\mathbf{E} = \rho/3\epsilon_0\,\mathbf{d}$.



Problem 1

2. **Griffiths 2.20** We'll calculate the curl of E, because a real electrostatic field must have zero curl. For (a) we have

$$\boldsymbol{\nabla}\times\mathbf{E} = -2kz\,\hat{\mathbf{x}} - 3kz\,\hat{\mathbf{y}} - kx\,\hat{\mathbf{z}} \neq 0,$$

   so (a) is *not* a possible electrostatic field. For (b),

$$\boldsymbol{\nabla}\times\mathbf{E} = k(2z-2z)\,\hat{\mathbf{x}} + (0-0)\,\hat{\mathbf{y}} + (2y-2y)\,\hat{\mathbf{z}} = \mathbf{0}$$

   Thus (b) is a possible electrostatic field. Now we want to compute the potential at some point (x_0, y_0, z_0), where the origin is at zero potential, using the relationship that $V = -\int \mathbf{E}\cdot d\ell$. Let's choose a simple path that goes in straight lines from $(0,0,0)$ to $(x_0,0,0)$ to $(x_0,y_0,0)$ to (x_0,y_0,z_0). There are three parts to the integral, and on each part we have a different $\mathbf{E}\cdot d\ell = ky^2\,dx + k(2xy+z^2)\,dy + 2kyz\,dz$. On the first segment, $y = z = dy = dz = 0$ so we get no contribution because $\mathbf{E}\cdot d\ell = 0$. On the second segment, $z = dz = dx = 0$ and $x = x_0$, so we get the contribution $\int \mathbf{E}\cdot d\ell = 2kx_0\int_0^{y_0} y\,dy = kx_0y_0^2$. On the final segment, $dx = dy = 0$ while $x = x_0$ and $y = y_0$, so we get $\int \mathbf{E}\cdot d\ell = 2ky_0\int_0^{z_0} z\,dz = ky_0z_0^2$. Now replacing x_0 with x and so on, we find $V(x,y,z) = -\int \mathbf{E}\cdot d\ell = -k(xy^2+yz^2)$. We can check that we did the integrals right by computing $\mathbf{E} = -\boldsymbol{\nabla}V = k\big(y^2\,\hat{\mathbf{x}} + (2xy+z^2)\,\hat{\mathbf{y}} + 2yz\,\hat{\mathbf{z}}\big)$, which gives us back the electric field we started with.

3. **Griffiths 2.25 (c)** We can use the second equation in (2.30) to calculate the potential due to a disk with uniform surface charge:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\mathbf{r}')}{r}\,da'.$$

Using cylindrical coordinates with angle ϕ and radius s, for this case $da' = \sigma s\, ds d\phi$ and $r = \sqrt{s^2+z^2}$.

$$V = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} d\phi \int_0^R \frac{\sigma s ds}{\sqrt{s^2+z^2}} = \frac{2\pi\sigma}{4\pi\epsilon_0}\left[\sqrt{s^2+z^2}\right]_0^R = \frac{\sigma}{2\epsilon_0}\left(\sqrt{R^2+z^2}-z\right)$$

Now, V is independent of x and y. Thus $\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0$, so

$$\mathbf{E} = -\boldsymbol{\nabla} V = -\frac{\partial V}{\partial z}\hat{\mathbf{z}} = -\frac{\sigma}{2\epsilon_0}\left[\frac{1}{2}\frac{2z}{\sqrt{R^2+z^2}} - 1\right] = \frac{z\sigma}{2\epsilon_0}\left[\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}}\right]\hat{\mathbf{z}}.$$

This is exactly what we got calculating the electric field directly in Problem Set 1.

4. **Griffiths 2.32 (a-b)** First we need to find the potential and electric field produced by a uniformly charged solid sphere of radius R and charge q. Outside the sphere, the electric field looks just like that from a point charge, $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$. Inside the sphere, we can use Gauss's Law: $\int \mathbf{E}\cdot d\mathbf{a} = 4\pi r^2 E(r) = Q_{\text{enc}}/\epsilon_0 = qr^3/(\epsilon_0 R^3) \Rightarrow E = \frac{1}{4\pi\epsilon_0}\frac{qr}{R^3}\hat{\mathbf{r}}$. To find the potential, we need to do a line integral of the electric field in from infinity. For $r > R$,

$$V(r) = -\int_\infty^r \frac{1}{4\pi\epsilon_0}\frac{q}{r'^2}dr' = \frac{1}{4\pi\epsilon_0}\frac{q}{r'}\bigg|_\infty^r = \frac{q}{4\pi\epsilon_0}\frac{1}{r}.$$

For $r < R$,

$$V(r) = -\int_\infty^R \frac{1}{4\pi\epsilon_0}\frac{q}{r'^2}dr' - \int_R^r \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}r'dr' = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{R} - \frac{1}{R^3}\left(\frac{r^2-R^2}{2}\right)\right] = \frac{q}{4\pi\epsilon_0}\frac{1}{2R}\left(3 - \frac{r^2}{R^2}\right)$$

(a) Using equation 2.43, with $\rho = \dfrac{q}{(4/3)\pi R^3}$ inside the sphere and zero outside, we have

$$W = \frac{1}{2}\int \rho V d\tau = \frac{1}{2}\frac{3q}{4\pi R^3}\int d\Omega \int_0^R \frac{q}{8\pi\epsilon_0 R}\left(3 - \frac{r^2}{R^2}\right) r^2\, dr = \frac{3q^2 4\pi}{64\pi^2\epsilon_0 R^4}\left[r^3 - \frac{r^5}{5R^2}\right]_0^R = \frac{1}{4\pi\epsilon_0}\left(\frac{3q^2}{5R}\right)$$

(b) Now using equation 2.45:

$$\begin{aligned} W &= \frac{\epsilon_0}{2}\int E^2 d\tau = \frac{\epsilon_0}{2}\left(\frac{q}{4\pi\epsilon_0}\right)^2 \int d\Omega \left\{\int_R^\infty \frac{1}{r^4}r^2 dr + \int_0^R \frac{r^2}{R^6}r^2\, dr\right\} \\ &= \frac{q^2}{8\pi\epsilon_0}\left\{\left[-\frac{1}{r}\right]_R^\infty + \left[\frac{r^5}{5R^6}\right]_0^R\right\} = \frac{1}{4\pi\epsilon_0}\left(\frac{3q^2}{5R}\right) \end{aligned}$$

Fortunately, both the solutions give the same result.

5. **Griffiths 2.36 (a-c)**

(a) For each of the cavities, we may imagine a Gaussian surface that is completely in the conductor and surrounds the cavity. Since there is no electric field in the metal of the conductor, $\int \mathbf{E}\cdot d\mathbf{a} = 0$. By Gauss's Law this means that the charge enclosed must be zero, so the total charge on the inner surface of the cavity must be exactly opposite of the point charge contained in the cavity. By symmetry, there is no reason for the surface charge to be anything but uniformly distributed. Thus $\sigma_a = q_a/4\pi a^2$ and $\sigma_b = q_b/4\pi b^2$. Since the conductor is neutral, the charge on the outer surface must be opposite the charge on the inner surface, and again it will be uniformly distributed, so $\sigma_R = (q_a+q_b)/4\pi R^2$.

(b) To find the field outside the conductor, the argument is exactly the same as in Example 2.9 in the text. The conductor makes the electric field outside look exactly like two point charges q_a and q_b at the origin. So $\mathbf{E} = \dfrac{1}{4\pi\epsilon_0}\dfrac{q_a+q_b}{r^2}\hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is a unit vector from the center of the conducting sphere.

(c) The surface charge in cavity a cancels the electric field due to the point charge q_a everywhere outside the cavity. So the only source of electric field in cavity a is the point charge q_a and the surface charge. But using Gauss's Law and the spherical symmetry, the electric field inside the cavity is just that of the point charge q_a, namely $\mathbf{E}_a = \dfrac{1}{4\pi\epsilon_0}\dfrac{q_a}{r_a^2}\,\hat{\mathbf{r}}_a$, where $\mathbf{r}_a$ is a unit vector from the center of cavity a. The same reasoning applies to cavity b, so $\mathbf{E}_b = \dfrac{1}{4\pi\epsilon_0}\dfrac{q_b}{r_b^2}\,\hat{\mathbf{r}}_b$.

6. **Griffiths 2.39** To find the capacitance between two conductors, we imagine placing a charge $+Q$ on one and $-Q$ on the other and then calculate the potential difference between them. Then we can find the capacitance from $C = Q/V$. (Note that the capacitance should depend only on the physical size of the system and not on the imaginary charge Q.)

   So in this case lets put a charge per unit length $+\lambda$ on the inner cylinder, and $-\lambda$ on the outer cylinder. To calculate the potential difference between the two cylinders, we need to integrate the electric field. Since the charge is evenly distributed, we can draw a Gaussian cylinder with radius r and length L between the two conductors. The electric field is pointing radially, and we can find its magnitude: $\int \mathbf{E}\cdot d\mathbf{a} = 2\pi rLE = Q_{\text{enc}}/\epsilon_0 = \lambda L/\epsilon_0 \Rightarrow E = \lambda/2\pi\epsilon_0 r$. The potential difference is then

$$V(b) - V(a) = -\int_a^b \mathbf{E}\cdot d\ell = -\frac{\lambda}{2\pi\epsilon_0}\int_a^b \frac{1}{r}dr = -\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right).$$

   Since the inner cylinder is at a higher potential (the potential drops in going from a to b), the positive voltage between the two conductors is $V = V(a) - V(b) = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right)$. Then $C = Q/V = \lambda L/V = \frac{2\pi\epsilon_0 L}{\ln\left(\frac{b}{a}\right)}$ so the capacitance per unit length is $C/L = \dfrac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}$. Note that this is independent of λ and Q.

7. **Griffiths 2.50** The differential form of Gauss's Law tells us $\nabla\cdot\mathbf{E} = \rho/\epsilon_0$. Thus in this case, we find $\rho = \epsilon_0\nabla\cdot\mathbf{E} = \epsilon_0 a$. This is a constant, uniform charge density. So why should the electric field point in the x-direction and not in the y-direction? In fact, it could, because you find exactly the same charge density for the fields $\mathbf{E} = ay\,\hat{\mathbf{y}}$ and $\mathbf{E} = (a/3)\mathbf{r}$. The point is that the *differential equations* $\nabla\cdot\mathbf{E} = \rho/\epsilon_0$ and $\nabla\times\mathbf{E} = 0$ are not sufficient to determine the electric field; boundary conditions are also necessary. It is just like asking for a function whose derivative is 3. There are many such possibilities: $f(x) = 3x$, $g(x) = 3x + 10$, $h(y) = 3y + c$; until you know some boundary conditions (such as $f(0) = 2$), you cannot give a unique answer. Knowing the field you can determine the charge distribution, but it doesn't work in reverse: knowing the charge distribution is not always enough to determine the field.

8. **Handout** We have two concentric spherical shells, and the outer one is being driven with a potential $\phi_2(t) = V_0\cos\omega t$. We make the approximation that the potential between the spheres is $\phi_2(t)$ everywhere. This would be true for the original version of Gauss's Law (i.e. for a zero mass photon), so call that solution the "original solution". The original solution is *not* an exact solution to the new equations, but since the change in the equations is very small, the original solution must be very close to the "new solution". So we will plug in the "original solution" to the new equation and see how much the "original solution" is modified. The error we make here is second order in the difference between the "original solution" and the "new solution", so it can be ignored for the purposes of determining the sensitivity required to carry out this experiment. We start with the given differential equation and integrate both sides over a sphere of radius $R_1 < r < R_2$.

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} - \frac{\phi_2}{\bar{\lambda}^2} \quad\Rightarrow\quad \int\nabla\cdot\mathbf{E}\,d\tau = \int\left[\frac{\rho}{\epsilon_0} - \frac{\phi_2}{\bar{\lambda}^2}\right]d\tau$$

   Now we can compute both sides of this equation separately. By symmetry, we assume that the electric field is purely radial and has the same magnitude all over our spherical surface. Thus $\int\mathbf{E}\cdot d\tau = 4\pi r^2 E(r)$. The second term on the right hand side is independent of r, and $\int\rho\,d\tau = Q_{\text{enc}}$. So we find, after substituting in the definition of $\bar{\lambda}$,

$$4\pi r^2 E(r) = \left[\frac{Q_{\text{enc}}}{\epsilon_0} - \frac{4}{3}\pi r^3 \frac{\phi_2}{\bar{\lambda}^2}\right] \quad \Rightarrow \quad \mathbf{E}(r) = \left[\frac{q}{4\pi\epsilon_0 r^2} - \frac{V_0 m_0^2 c^2}{3\hbar^2} r \cos\omega t\right] \hat{\mathbf{r}}.$$

Here $Q_{\text{enc}} = q$, the charge on the inner sphere.

Now, to calculate the potential difference we integrate the electric field.

$$V(R_2) - V(R_1) = -\int_{R_1}^{R_2} \mathbf{E} \cdot d\ell = -\int_{R_1}^{R_2} E(r) dr = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R_2} - \frac{1}{R_1}\right) + \frac{V_0 m_0^2 c^2}{3\hbar^2} \cos(\omega t) \frac{1}{2}\left(R_2^2 - R_1^2\right)$$

Now lets plug in some numbers. The wording is a little confusing, since V_0 is the amplitude of ϕ_2, it can't really be the peak-t0-peak voltage. So I'll just take $V_0 = 10$ kV and compute the amplitude of the measured voltage. If you did something slightly different, that fine. The other numbers are $q = 0$, $R_1 = 0.5$ m, $R_2 = 1.5$ m, and $m_0 = 10^{-15}$ eV/c^2. Be careful about converting all the units. I get $\Delta V = 8.53 \times 10^{-14} \cos\omega t$ volts. Anything between half or twice this value is acceptable.

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Problem Set 3

1. Griffiths 3.1

2. Griffiths 3.4

3. Griffiths 3.6

4. Griffiths 3.9

5. Griffiths 3.10

6. Griffiths 3.14

7. Griffiths 3.15

8. Griffiths 3.43

University of California, Berkeley
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Solution Set 3 (compiled by Daniel Larson)

1. **Griffiths 3.1** We want to calculate the average potential on the surface of a sphere due to a point charge q located somewhere within the sphere. Define our coordinates so that the sphere of radius R is centered at the origin and the point charge lies on the z-axis a distance z from the origin. This calculation is identical to the one on page 114 of the text, except that $z < R$, so when it comes time to evaluate the integral we will get a term $\sqrt{(z-R)^2} = |z-R| = R-z$.

   At any point on the sphere, the potential is $V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$ where $r^2 = R^2 + z^2 - 2Rz\cos\theta$ (see Figure 3.3 for the setup, but imagine $z < R$). We need to calculate

$$\begin{aligned}
V_{\text{ave}} &= \frac{1}{4\pi R^2}\frac{q}{4\pi\epsilon_0}\int \frac{R^2\sin\theta\, d\theta\, d\phi}{\sqrt{z^2+R^2-2Rz\cos\theta}} = \frac{1}{4\pi\epsilon_0}\frac{q}{2Rz}\left.\sqrt{z^2+R^2-2Rz\cos\theta}\right|_0^\pi \\
&= \frac{1}{4\pi\epsilon_0}\frac{q}{2Rz}\left(\sqrt{z^2+R^2+2Rz} - \sqrt{z^2+R^2-2Rz}\right) = \frac{1}{4\pi\epsilon_0}\frac{q}{2Rz}\left(\sqrt{(z+R)^2} - \sqrt{(z-R)^2}\right) \\
&= \frac{1}{4\pi\epsilon_0}\frac{q}{2Rz}(z+R-(R-z)) = \frac{q}{4\pi\epsilon_0 R}
\end{aligned}$$

   Note the term $R-z$ as mentioned above, since $z < R$ for a charge inside the sphere. Also notice that the above result doesn't depend on the exact location of the point charge. Thus if there were more than one charge, we would find $V_{\text{ave}} = \frac{Q_{\text{enc}}}{4\pi\epsilon_0 R}$. Putting this together with the result in the text for charges outside the sphere, we have

$$V_{\text{ave}} = V_{\text{center}} + \frac{Q_{\text{enc}}}{4\pi\epsilon_0 R}$$

2. **Griffiths 3.4** We have a region of space enclosed by one or more boundaries, the charge density ρ is given inside the region, and *either* V or $\frac{\partial V}{\partial n}$ is specified on each boundary. (The situation is much like Figure 3.6 in the text, but we're not assuming any surface is a conductor.) To prove that a solution is unique, we assume that there are two different solutions and then show that they must be equal. So assume that there are two different electric fields $\mathbf{E}_1$ and $\mathbf{E}_2$ in the region that satisfy

$$\boldsymbol{\nabla}\cdot\mathbf{E}_1 = \frac{\rho}{\epsilon_0} = -\nabla^2 V_1 \qquad\qquad \boldsymbol{\nabla}\cdot\mathbf{E}_2 = \frac{\rho}{\epsilon_0} = -\nabla^2 V_2$$

   Now let $\mathbf{E}_3 = \mathbf{E}_1 - \mathbf{E}_2$ and $\mathbf{E}_3 = -\boldsymbol{\nabla}V_3 = -\boldsymbol{\nabla}(V_1 - V_2)$. Subtracting the above equations we find $\boldsymbol{\nabla}\cdot\mathbf{E}_3 = \boldsymbol{\nabla}\cdot\mathbf{E}_1 - \boldsymbol{\nabla}\cdot\mathbf{E}_2 = 0$. Then

$$\boldsymbol{\nabla}\cdot(V_3\mathbf{E}_3) = V_3(\boldsymbol{\nabla}\cdot\mathbf{E}_3) + \mathbf{E}_3\cdot(\boldsymbol{\nabla}V_3) = \mathbf{E}_3\cdot(-\mathbf{E}_3) = -(E_3)^2$$

   Now using the divergence theorem on the above equation for a surface S_i that encloses a volume $\mathcal{V}_i$, we have:

$$\int_{S_i} V_3\mathbf{E}_3\cdot d\mathbf{a} = -\int_{\mathcal{V}_i}(E_3)^2\, d\tau. \tag{1}$$

   Now there are two cases. (I) If the potential is specified on the surface S_i, then we must have the two different potentials agree there, namely $V_1(S_i) = V_2(S_i)$, which means

$$\int_{S_i} V_3\mathbf{E}_3\cdot d\mathbf{a} = \int_{S_i}(V_1 - V_2)\mathbf{E}_3\cdot d\mathbf{a} = 0.$$

   (II) If the normal derivative $\frac{\partial V}{\partial n} = \boldsymbol{\nabla}V\cdot\hat{\mathbf{n}}$ is specified on the surface S_i, then we must have $\frac{\partial V_1}{\partial n}(S_i) = \frac{\partial V_2}{\partial n}(S_i)$, so

$$\int_{S_i} V_3\mathbf{E}_3\cdot d\mathbf{a} = -\int_{S_i} V_3\boldsymbol{\nabla}(V_1 - V_2)\cdot\hat{\mathbf{n}}da = -\int_{S_i} V_3(\boldsymbol{\nabla}V_1\cdot\hat{\mathbf{n}} - \boldsymbol{\nabla}V_2\cdot\hat{\mathbf{n}})da = -\int_{S_i} V_3\left(\frac{\partial V_1}{\partial n} - \frac{\partial V_2}{\partial n}\right)da = 0.$$

But looking back at equation (1) above, we see that both cases imply

$$0 = \int_{\mathcal{V}_i} (\mathbf{E}_3)^2 \, d\tau = \int_{\mathcal{V}_i} |\mathbf{E}_1 - \mathbf{E}_2|^2 .$$

If we do the integral over all the surfaces in the region, the volume $\mathcal{V}_i$ is simply the total volume of the region. Since the integrand, $|\mathbf{E}_1 - \mathbf{E}_2|^2 \geq 0$, the only way the above equation can hold is if the integrand is in fact equal to zero, which means $\mathbf{E}_1 = \mathbf{E}_2$. Thus the field is uniquely determined if the charge density is given everywhere and either V or $\frac{\partial V}{\partial n}$ is specified on each boundary.

3. **Griffiths 3.6** The xy plane is a grounded conductor, so it is at zero potential. We can reproduce this situation by considering a similar setup without the conductor, but instead with a charge $+2q$ at $z = -d$ and a charge $-q$ at $z = -3d$. These image charges make the potential $V = 0$ anywhere in the xy plane, so it exactly matches the boundary conditions in the original problem with the conductor. The force on the charge $+q$ is then given by Coulomb's Law:

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\left(\frac{(q)(-q)}{(6d)^2} + \frac{(q)(2q)}{(4d)^2} + \frac{(q)(-2q)}{(2d)^2}\right)\hat{\mathbf{z}} = \frac{1}{4\pi\epsilon_0}\left(\frac{29q^2}{72d^2}\right)\hat{\mathbf{z}}.$$

4. **Griffiths 3.9** Again, we want to find some image charges that give $V = 0$ in the xy plane. So we put a uniform line charge $-\lambda$ parallel to the x-axis and a distance d directly *below* it.

   (a) The potential due to a single infinite line charge is $V(r) = -\frac{2\lambda}{4\pi\epsilon_0}\ln(r/r_0)$ where r is the perpendicular distance to the line charge and r_0 is an arbitrary reference distance. Let's choose the reference distance to be d for both the positive and negative line charges; this automatically gives zero potential on the xy-plane. We want to find the potential at an arbitrary point in the yz plane (the potential must be independent of x because of translational symmetry in the x-direction). Let s_+ and s_- be the perpendicular distance between the point $P = (y, z)$ and the positive and negative line charges. The potential at P is the sum of the potentials due to each line charge:

$$V(y,z) = \frac{2\lambda}{4\pi\epsilon_0}\left(\ln\frac{s_-}{d} - \ln\frac{s_+}{d}\right) = \frac{2\lambda}{4\pi\epsilon_0}\ln\frac{s_-}{s_+} = \frac{\lambda}{4\pi\epsilon_0}\ln\frac{s_-^2}{s_+^2} = \frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{y^2+(z+d)^2}{y^2+(z-d)^2}\right)$$

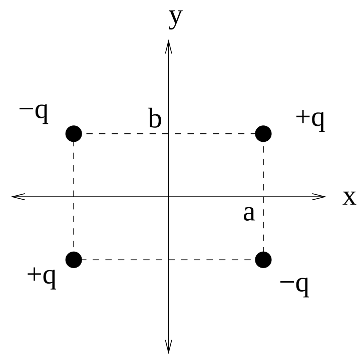
   We can check our result by verifying that $V(z = 0) = 0$ as it must, since the conductor in the xy plane is grounded.

   (b) To find the charge density on the conducting plane of the original problem, we make use of Equation 2.49. In this case the normal to the xy plane is in the z direction.

$$\sigma(y) = -\epsilon_0 \frac{\partial V}{\partial n}\bigg|_{z=0} = -\epsilon_0 \frac{\partial V}{\partial z}\bigg|_{z=0} = -\epsilon_0 \frac{\lambda}{4\pi\epsilon_0}\left(\frac{2(z+d)}{y^2+(z+d)^2} - \frac{2(z-d)}{y^2+(z-d)^2}\right)\bigg|_{z=0} = -\frac{\lambda d}{\pi(y^2+d^2)}$$

5. **Griffiths 3.10** We want to find the potential in the first quadrant, so we are only allowed to add image charges outside this region. We can add an image charge $-q$ at $(x, y) = (a, -b)$ to give zero potential along the x-axis. To get zero potential along the y-axis we need to add *two more* image charges to balance the two charges we have already. They should have opposite charge and be placed as shown in the figure below. Assume all the charges lie in the xy plane. The potential is the sum of the contributions from the four charges:

$$\begin{aligned} V(x,y,z) &= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{(x-a)^2+(y-b)^2+z^2}} + \frac{q}{\sqrt{(x+a)^2+(y+b)^2+z^2}}\right. \\ &\qquad \left. - \frac{q}{\sqrt{(x+a)^2+(y-b)^2+z^2}} - \frac{q}{\sqrt{(x-a)^2+(y+b)^2+z^2}}\right] \end{aligned}$$



Problem 5. Griffiths 3.10

The force on q due to the conducting planes is the same as the force on q due to the image charges, which is a sum of three contributions. But we need to remember that the force is a vector and keep track of all three components. First of all, since the charges all lie in the xy plane, there is no z-component: $F_z = 0$. The other components follow from Coulomb's law and breaking the force vectors into components.

$$F_x = \frac{1}{4\pi\epsilon_0}\left[\frac{-q^2}{4a^2} + \frac{q^2}{4(a^2+b^2)}\frac{a}{\sqrt{a^2+b^2}}\right] \qquad F_y = \frac{1}{4\pi\epsilon_0}\left[\frac{-q^2}{4b^2} + \frac{q^2}{4(a^2+b^2)}\frac{b}{\sqrt{a^2+b^2}}\right]$$

The easiest way to find the work needed to bring the charge q in from infinity into the corner made by the conducting planes is to compute the total work needed to bring together the collection of image charges and then divide by 4, because we don't count the work needed to bring in the image charges, for in the original problem the only other charges present are those induced in the conductors, but the induced charge comes "for free" because conductors are equipotential surfaces. Thus the work to bring in the single charge q is (using equation 2.40 and multiplying by 1/4):

$$W = \frac{1}{4}\frac{1}{4\pi\epsilon_0}\left(-\frac{q^2}{2a} - \frac{q^2}{2b} + \frac{q^2}{2\sqrt{a^2+b^2}} - \frac{q^2}{2a} - \frac{q^2}{2b} + \frac{q^2}{2\sqrt{a^2+b^2}}\right) = -\frac{q^2}{16\pi\epsilon_0}\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{\sqrt{a^2+b^2}}\right)$$

This method would work for any angle which evenly divides 360°, namely $360°/2n$ for $n = 1, 2, 3, \ldots$.

6. **Griffiths 3.14**

(a) In this problem, the pipe is infinite in the z-direction, so there can be no dependence on z because of the symmetry. Thus we are left with solving Laplace's equation in two dimensions:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Using separation of variables, we assume $V(x,y) = X(x)Y(y)$. Plugging into the above equation, and letting primes denote derivatives of single variable functions with respect to their argument, we get

$$Y(y)X''(x) + X(x)Y''(y) = 0 \quad \Rightarrow \quad \frac{1}{X}X''(x) + \frac{1}{Y}Y''(y) = 0$$

In order fo this equation to hold for all x and y, we must have both terms equal to constants. Since the potential must vanish at $y = 0$ and $y = a$, it makes sense to use sines and cosines in the y-direction, which means we want to put a negative constant in the y-equation.

$$\frac{1}{Y}Y''(y) = -k^2 \qquad \frac{1}{X}X''(x) = k^2 \quad \text{for } k \text{ constant}$$

The solutions for the y equation give $Y(y) = A\sin ky + B\cos ky$. For the x equation, we need it to vanish at $x = 0$, so lets choose hyperbolic trig functions instead of exponentials: $X(x) = C\sinh kx + D\cosh kx$.

Thus $V(x,y) = (A\sin ky + B\cos ky)(C\sinh kx + D\cosh kx)$. Now we need to choose the coefficients A, B, C, D to satisfy the boundary conditions. $V(x, y=0) = 0$ means we need $B = 0$. $V(x=0, y) = 0$ means we need $D = 0$. $V(x, y=a) = 0$ means we need $\sin ka = 0 \Rightarrow ka = n\pi$ for $n = 1, 2, 3, \ldots$. The most general solution at this stage is a linear combination of solutions for different n, where I've combined the constants A and C into A_n:

$$V(x,y,z) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$

To determine the A_n we need the last boundary condition, $V(x=b,y) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi y}{a}\right) = V_0(y)$. Using Fourier's trick, we multiply both sides by $\sin\left(\frac{m\pi y}{a}\right)$ and integrate from 0 to a. This gives

$$A_m \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a V_0(y)\sin\left(\frac{m\pi y}{a}\right)dy \quad\Rightarrow\quad A_n = \frac{2}{a\sinh(n\pi b/a)}\int_0^{\infty} V_0(y)\sin\left(\frac{n\pi y}{a}\right)dy$$

Thus the equation for $V(x,y,z)$ together with the formula for A_n gives a general formula for the potential within the pipe.

(b) With $V_0(y) = V_0 =$ constant, we find

$$A_n = \frac{2}{a\sinh(n\pi b/a)}V_0\int_0^{\infty}\sin\left(\frac{n\pi y}{a}\right)dy = \frac{2V_0}{a\sinh(n\pi b/a)}\begin{cases} 0, & \text{if } n \text{ is even,}\\ \frac{2a}{n\pi}, & \text{if } n \text{ is odd.}\end{cases}$$

So we find

$$V(x,y) = \frac{4V_0}{\pi}\sum_{n=1,3,5,\ldots}\frac{\sinh(n\pi x/a)\sin(n\pi y/a)}{n\sinh(n\pi b/a)}.$$

7. **Griffiths 3.15** Another problem where we need to use separation of variables, but this time with all three dimensions. Proceeding as before, we assume $V(x,y,z) = X(x)Y(y)Z(z)$ and plug this into Laplace's equation, to find

$$\frac{1}{X}X''(x) + \frac{1}{Y}Y''(y) + \frac{1}{Z}Z''(z) = 0.$$

Each of these terms must be constant, and the sum of the three constants must be zero. We want to choose the constants appropriately by looking at the boundary conditions. In the x and y directions there are grounded plates at 0 and a, which means the solutions will be sines and cosines in those directions, so we choose the constants for the X and Y terms to be negative. In order to add to zero, the other constant must be positive.

$$\frac{1}{X}X''(x) = -k^2 \qquad \frac{1}{Y}Y''(y) = -l^2 \qquad \frac{1}{Z}Z''(z) = k^2 + l^2 \quad \text{for } k,\ l \text{ constants}$$

Now we can write down the general solutions to these equations. Since Z must vanish at $z = 0$, it is easier to write down the solution in terms of hyperbolic trig functions instead of real exponentials.

$$X(x) = A\sin kx + B\cos kx, \quad Y(y) = C\sin ly + D\cos ly, \quad Z(z) = E\sinh(z\sqrt{k^2+l^2}) + F\cosh(z\sqrt{k^2+l^2})$$

The boundary conditions tell us $V(0,y,z) = V(x,0,z) = V(x,y,0)$, so to make this hold for all values of the other variables, we must have $B = D = F = 0$. Then $V(a,y,z) = V(x,a,z) = 0$ requires $k = n\pi/a$ and $l = m\pi/a$ for positive integers n and m. So at this stage, the most general solution is a linear combination of solutions for all n and m.

$$V(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{n,m}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\sinh\left(\frac{\pi z\sqrt{n^2+m^2}}{a}\right)$$

The only thing that remains is to fix the constants $A_{n,m}$ by using the last boundary condition: $V(x,y,a) = V_0$. Using Fourier's trick, we set $z = a$, multiply both sides by $\frac{2}{a}\sin(n'\pi x/a)\frac{2}{a}\sin(m'\pi y/a)$ and integrate over both x and y from 0 to a. This will pick out the coefficient $A_{n',m'}$.

$$A_{n',m'}\sinh(\pi\sqrt{n'^2+m'^2}) = \left(\frac{2}{a}\right)^2 V_0\int_0^a\int_0^a \sin\left(\frac{n'\pi x}{a}\right)\sin\left(\frac{m'\pi y}{a}\right)dx\,dy = \begin{cases} 0, & \text{if } n' \text{ or } m' \text{is even,}\\ \frac{16V_0}{\pi^2 n'm'}, & \text{if both are odd.}\end{cases}$$

The above equation gives us $A_{n,m}$, which we can plug into the double sum above. The final solution is

$$V(x,y,z) = \frac{16V_0}{\pi^2} \sum_{\text{odd } n} \sum_{\text{odd } m} \frac{1}{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) \frac{\sinh(\pi z\sqrt{n^2+m^2}/a)}{\sinh(\pi\sqrt{n^2+m^2})}.$$

8. **Griffiths 3.43**

(a) To use the hint, we need to figure out what it means to integrate by parts in three dimensions. We can work it out starting from vector identity (5) in the front cover of Griffiths. With a scalar function V and a vector function $\mathbf{E}$ the identity can be written

$$\mathbf{E} \cdot (\boldsymbol{\nabla} V) = \boldsymbol{\nabla} \cdot (V\mathbf{E}) - V(\boldsymbol{\nabla} \cdot \mathbf{E}).$$

Now we integrate both sides over a volume $\mathcal{V}$ with surface S and use the divergence theorem on the first term on the right hand side. This yields a formula for three-dimensional integration by parts:

$$\int_{\mathcal{V}} \mathbf{E} \cdot (\boldsymbol{\nabla} V)\, d\tau = \int_S V\mathbf{E} \cdot d\mathbf{a} - \int_{\mathcal{V}} V(\boldsymbol{\nabla} \cdot \mathbf{E})\, d\tau$$

Now assume we have two completely different systems, numbered 1 and 2, each of which has a certain charge density ρ, potential V and electric field $\mathbf{E}$. Following the hint we will integrate $\mathbf{E}_1 \cdot \mathbf{E}_2$ in two ways.

$$\int_{\mathcal{V}} \mathbf{E}_1 \cdot \mathbf{E}_2 \; d\tau = -\int_{\mathcal{V}} (\boldsymbol{\nabla} V_1) \cdot \mathbf{E}_2\, d\tau = -\int_S V_1 \mathbf{E}_2 \cdot \, d\mathbf{a} + \int_{\mathcal{V}} V_1 (\boldsymbol{\nabla} \cdot \mathbf{E}_2)\, d\tau = -\int_S V_1 \mathbf{E}_2 \cdot \, d\mathbf{a} + \int_{\mathcal{V}} V_1 \rho_2/\epsilon_0\, d\tau$$

If we assume that the charge distributions are localized (i.e. do not extend to infinity) then we can take our volume to be all of space, which means that the surface S is at infinity, where the potential V_1 falls off to zero. So the surface integral vanishes, leaving

$$\int_{\mathcal{V}} \mathbf{E}_1 \cdot \mathbf{E}_2 \; d\tau = \frac{1}{\epsilon_0} \int_{\mathcal{V}} V_1 \rho_2\, d\tau$$

We can do exactly the same manipulations after replacing $\mathbf{E}_2$ with $-\boldsymbol{\nabla} V_2$, so we'll arrive at the same result with the labels 1 and 2 switched. So we conclude

$$\epsilon_0 \int_{\mathcal{V}} \mathbf{E}_1 \cdot \mathbf{E}_2 \; d\tau = \int_{\mathcal{V}} V_1 \rho_2\, d\tau = \int_{\mathcal{V}} V_2 \rho_1\, d\tau$$

(b) Now we want to apply the above result to a specific situation. It will be less confusing if I call the conductors a and b instead of 1 and 2. In the first system we have two conductors and we put a charge Q on conductor a, and let V_{ab} be the potential at conductor b. So in this system ρ_1 is zero everywhere except on conductor a, where there is total charge Q distributed in some complicated way. But this means that $\int_{\mathcal{V}} \rho_1\, d\tau = Q$. The potential in this system is complicated. The only place we know what it is is on conductor b, where V_1 is a constant, $V_1 = V_{ab}$.

Now consider the second system. It consists of the same two conductors a and b in the same positions, but this time we put charge Q on conductor b and call the potential at conductor a V_{ba}. Here, ρ_2 is zero everywhere except on conductor b. But we know $\int_{\mathcal{V}} \rho_2\, d\tau = Q$. The potential is complicated, and all we know is that on conductor a it is constant and equal to V_{ba}.

Now we apply Green's reciprocity theorem. When we calculate $\int \rho_1 V_2\, d\tau$, ρ_1 vanishes everywhere except conductor a, but that is exactly where we know what V_2 is; it's a constant equal to V_{ba}. Thus

$$\int \rho_1 V_2\, d\tau = V_{ba} \int \rho_1\, d\tau = V_{ba} Q.$$

But we also have

$$\int \rho_2 V_1\, d\tau = V_{ab} \int \rho_2\, d\tau = V_{ab} Q,$$

where again we could do the integral because ρ_2 is zero everywhere except on conductor b where $V_1 = V_{ab}$. The Q's cancel, leaving us with the result $V_{ab} = V_{ba}$, or in the notation of the problem, $V_{12} = V_{21}$.

University of California, Berkeley
Physics 110A Fall 2001   Section 1 (*Strovink*)

Problem Set 4

1. Griffiths 3.33

2. Griffiths 4.4

3. Griffiths 4.6

4. Griffiths 4.10

5. Griffiths 4.13. [*Hint:* Consider the uniformly polarized cylinder to be the superposition of two cylinders that are uniformly charged throughout their volume, one positive, the other negative, with a small relative offset. Take the limit as the offset vanishes while the product of offset and charge remains finite.]

6. Griffiths 4.15

7. Griffiths 4.16

8. Griffiths 4.18

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Physics 110A Fall 2001 Section 1 (*Strovink*)

Solution Set 4 (compiled by Daniel Larson)

1. **Griffiths 3.33** To get a general formula for the electric field from an electric dipole, let's start with the general formula for the potential, (3.99) in the text.

$$\mathbf{E}_{\text{dip}}(\mathbf{r}) = -\nabla V_{\text{dip}}(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0}\nabla\left(\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}\right) = -\frac{1}{4\pi\epsilon_0}\nabla\left(\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\right) = -\frac{1}{4\pi\epsilon_0}\left[\frac{1}{r^3}\nabla(\mathbf{p}\cdot\mathbf{r}) + (\mathbf{p}\cdot\mathbf{r})\nabla\left(\frac{1}{r^3}\right)\right]$$

To evaluate the first term we use some vector identities.

$$\nabla(\mathbf{p}\cdot\mathbf{r}) = \mathbf{p}\times(\nabla\times\mathbf{r}) + \mathbf{r}\times(\nabla\times\mathbf{p}) + (\mathbf{p}\cdot\nabla)\mathbf{r} + (\mathbf{r}\cdot\nabla)\mathbf{p} = (\mathbf{p}\cdot\nabla)\mathbf{r},$$

because $\nabla\times\mathbf{r} = 0$ and $\mathbf{p}$ is a constant vector, so any derivatives of it vanish. To evaluate the one remaining term we can temporarily choose cartesian coordinates:

$$(\mathbf{p}\cdot\nabla)\mathbf{r} = \left(p_x\frac{\partial}{\partial x} + p_y\frac{\partial}{\partial y} + p_z\frac{\partial}{\partial z}\right)(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}) = p_x\,\hat{\mathbf{x}} + p_y\,\hat{\mathbf{y}} + p_z\,\hat{\mathbf{z}} = \mathbf{p}.$$

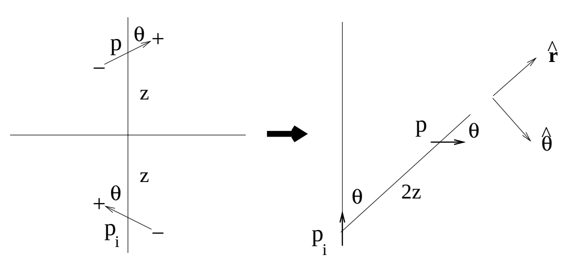
For the second term we need $\nabla r^n = nr^{n-1}\,\hat{\mathbf{r}}$. Putting the results together,

$$\mathbf{E}_{\text{dip}}(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0}\left[\frac{1}{r^3}\mathbf{p} + (\mathbf{p}\cdot\hat{\mathbf{r}})r\left(\frac{-3}{r^4}\right)\hat{\mathbf{r}}\right] = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{p}]$$

2. **Griffiths 4.4** The point charge produces an electric field with magnitude $E_1 = q/4\pi\epsilon_0 r^2$ at the location of the neutral atom. That electric field polarizes the atom, giving it a dipole moment $\mathbf{p} = \alpha\mathbf{E}_1 = -\alpha q/4\pi\epsilon_0 r^2\,\hat{\mathbf{r}}$ where $\mathbf{r}$ is the vector pointing from the atom towards the point charge. But the polarized atom produces its own field due to its dipole moment $\mathbf{p}$. At the location of the point charge, the electric field is $\mathbf{E}_2 = 2\mathbf{p}/4\pi\epsilon_0 r^3$ where I've used the result of the previous problem. Finally, the force felt by the point charge is attractive:

$$\mathbf{F} = q\mathbf{E}_2 = -\frac{\alpha q^2}{8\pi^2\epsilon_0^2 r^5}\,\hat{\mathbf{r}} = -2\alpha\left(\frac{q}{4\pi\epsilon_0}\right)^2\frac{1}{r^5}\,\hat{\mathbf{r}}.$$

3. **Griffiths 4.6** To determine the effect of the conducting plane, we can use an image dipole situated below the plane. To figure out how it should be pointing, we can think of the perfect dipole as two charges separated by a small distance, figure out where the image charges should be, and then let the distance between the charges in each dipole go to zero. This is shown in the figure.



The image dipole, $\mathbf{p}_i$, creates an electric field $\mathbf{E}_i$ at the position of the real dipole, which causes a torque on the real dipole, $\mathbf{N} = \mathbf{p}\times\mathbf{E}_i$. If we choose a coordinate system centered on the image dipole with $\mathbf{p}_i$ pointing in the z-direction, then the real dipole can be taken to be in the xz-plane with coordinates $(r,\theta,\phi) = (2z,\theta,0)$. Using equation (3.103) in the text, the electric field there is $\mathbf{E}_1 = \frac{p_i}{4\pi\epsilon_0(2z)^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$. Now, in order to take the cross product, we need to resolve $\mathbf{p}$ in the $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ directions. From the figure we see that $\mathbf{p}$ makes an angle θ with the $\hat{\mathbf{r}}$ direction. Thus $\mathbf{p} = p\cos\theta\,\hat{\mathbf{r}} + p\sin\theta\,\hat{\boldsymbol{\theta}}$.

$$\mathbf{N} = \mathbf{p}\times\mathbf{E}_i = (p\cos\theta\,\hat{\mathbf{r}} + p\sin\theta\,\hat{\boldsymbol{\theta}})\times\frac{p_i}{4\pi\epsilon_0(2z)^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}) = \frac{-p^2\cos\theta\sin\theta}{4\pi\epsilon_0(2z)^3}\,\hat{\boldsymbol{\phi}} = -\frac{1}{4\pi\epsilon_0}\frac{p^2\sin(2\theta)}{16z^3}\,\hat{\boldsymbol{\phi}}$$

where we have used $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ and $p_i = p$. Note that $-\hat{\boldsymbol{\phi}}$ is the direction out of the page. The torque vanishes for $\theta = 0, \pi/2$, and π. However, since the dipole wants to rotate one way for $0 < \theta < \pi/2$ and the other way for $\pi/2 < \theta < \pi$, at $\pi/2$ the torque is changing sign and so the dipole is not stable at that angle. The stable orientations are for $\theta = 0$ or π where the dipole is perpendicular to the conducting plane, pointing either toward or away from it.

4. **Griffiths 4.10**

   (a) $\sigma_b = \mathbf{P}(R) \cdot \hat{\mathbf{n}} = kR\,\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = kR. \qquad \rho_b = -\boldsymbol{\nabla} \cdot \mathbf{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 kr) = -\frac{1}{r^2} 3kr^2 = -3k.$

   (b) There is no free charge specified, and the sphere is not connected to any wires or batteries, so the free charge is zero everywhere. Thus the only charges contributing to the electric field are the bound charges. Because of the symmetry we know that the electric field can only be pointing radially, so we can use Gauss's law to find the field. Inside the sphere we make a gaussian sphere of radius r: $E(r)4\pi r^2 = \frac{4}{3}\pi r^3 \rho_b/\epsilon_0 \;\Rightarrow\; \mathbf{E}_{\text{in}}(r) = \rho_b r/3\epsilon_0\,\hat{\mathbf{r}} = -k/\epsilon_0\,\mathbf{r}$. Outside the sphere the total volume charge is $-3k\frac{4}{3}\pi R^3 = -4\pi R^3 k$ while the total surface charge is $kR \times 4\pi R^2 = 4\pi R^3 k$. Thus the net charge inside a Gaussian surface with radius $r > R$ is zero, which means $\mathbf{E}_{\text{out}} = \mathbf{0}$.

5. **Griffiths 4.13** We want to tackle this problem in exactly the same way we did the sphere with uniform polarization in class, or in example 4.3 in the text. We can think of the uniformly polarized cylinder as two cylinders with opposite uniform charge density $\pm\rho$ separated from each other by a small distance d. Start by considering a single, uniformly charged cylinder. Using Gauss's law we can find the electric field both outside and inside the cylinder. Using Griffiths's notation with $\mathbf{s}$ as the radial coordinate, inside the cylinder we find: $E2\pi s\ell = \frac{1}{\epsilon_0}\rho\pi s^2\ell \Rightarrow \mathbf{E} = (\rho/2\epsilon_0)\mathbf{s}$. In the region of overlap between the two cylinders we have contributions to $\mathbf{E}$ from both cylinders which add just like in Problem 2.18 (see solution set 2 for a figure). *However*, in this case we'll define $\mathbf{d}$ to be the vector pointing from the center of the negative cylinder to the center of the positive cylinder. Thus the total electric field in the region of overlap is $\mathbf{E} = -(\rho/2\epsilon_0)\mathbf{d}$. We can think of the two uniformly charged cylinders as being line charges with charge per length $\pm\lambda = \pm\pi a^2\rho$, which is like a bunch of dipoles $\lambda d\ell\mathbf{d}$ all in a row. Now thinking back to the single, polarized cylinder, the total dipole moment in a piece of length ℓ is $\mathbf{P}(\pi a^2\ell) = \lambda\ell\mathbf{d} = \pi a^2\rho\ell\mathbf{d}$, so $\mathbf{P} = \rho\mathbf{d}$. Plugging this into our expression for $\mathbf{E}$ we find $\mathbf{E}_{\text{in}} = -\frac{1}{2\epsilon_0}\mathbf{P}$.

   Now we need the electric field outside the cylinders. This time, for a single uniformly charged cylinder and $s > a$ Gauss's law gives: $E2\pi s\ell = \frac{1}{\epsilon_0}\pi a^2\ell\rho \Rightarrow \mathbf{E} = (\rho a^2/2\epsilon_0 s)\,\hat{\mathbf{s}}$. At some point outside the cylinders, let $\mathbf{s}_+$ and $\mathbf{s}_-$ be the radial vectors from the centers of the two charged cylinders to the point in question. The total electric field at that point gets contributions from both cylinders, so

$$\mathbf{E}_{\text{out}} = \mathbf{E}_+ + \mathbf{E}_- = \frac{\rho a^2}{2\epsilon_0}\left(\frac{\hat{\mathbf{s}}_+}{s_+} - \frac{\hat{\mathbf{s}}_-}{s_-}\right) = \frac{\rho a^2}{2\epsilon_0}\left(\frac{\mathbf{s}_+}{s_+^2} - \frac{\mathbf{s}_-}{s_-^2}\right)$$

   We want to simplify this expression, using the fact that $\mathbf{s}_+ - \mathbf{s}_- = -\mathbf{d}$ and $d \ll s_+, s_-$. Let $\mathbf{s}$ be the radial vector from the midpoint between the two charged cylinders; this is the true center of the uniformly polarized cylinder. Then $\mathbf{s}_\pm = \mathbf{s} \mp \frac{\mathbf{d}}{2}$.

$$\begin{aligned}\frac{\mathbf{s}_\pm}{s_\pm^2} &= \left(\mathbf{s} \mp \frac{\mathbf{d}}{2}\right)\left(s^2 \mp \mathbf{s}\cdot\mathbf{d} + \frac{d^2}{4}\right)^{-1} = \left(\mathbf{s} \mp \frac{\mathbf{d}}{2}\right)\frac{1}{s^2}\left(1 \mp \frac{\mathbf{s}\cdot\mathbf{d}}{s^2} + \frac{d^2}{4s^2}\right)^{-1} \\ &\approx \frac{1}{s^2}\left(\mathbf{s} \mp \frac{\mathbf{d}}{2}\right)\left(1 \pm \frac{\mathbf{s}\cdot\mathbf{d}}{s^2}\right) = \frac{1}{s^2}\left(\mathbf{s} \pm \mathbf{s}\frac{\mathbf{s}\cdot\mathbf{d}}{s^2} \mp \frac{\mathbf{d}}{2}\right)\end{aligned}$$

   where we have kept only the terms linear in the small quantity d/s. Using this result in the expression for the electric field, and the result that $\mathbf{P} = \rho\mathbf{d}$, we find

$$\mathbf{E}_{\text{out}} = \frac{\rho a^2}{2\epsilon_0}\frac{1}{s^2}\left[\left(\mathbf{s} + \mathbf{s}\frac{\mathbf{s}\cdot\mathbf{d}}{s^2} - \frac{\mathbf{d}}{2}\right) - \left(\mathbf{s} - \mathbf{s}\frac{\mathbf{s}\cdot\mathbf{d}}{s^2} + \frac{\mathbf{d}}{2}\right)\right] = \frac{\rho a^2}{2\epsilon_0}\frac{1}{s^2}\left(\frac{2\mathbf{s}(\mathbf{s}\cdot\mathbf{d})}{s^2} - \mathbf{d}\right) = \frac{a^2}{2\epsilon_0 s^2}[2(\mathbf{P}\cdot\hat{\mathbf{s}})\,\hat{\mathbf{s}} - \mathbf{P}]$$

6. **Griffiths 4.15**

   (a)

$$\rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{k}{r}\right) = -\frac{k}{r^2}; \quad \sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \begin{cases} \mathbf{P}\cdot\hat{\mathbf{r}} = k/b & (\text{at } r=b), \\ \mathbf{P}\cdot(-\hat{\mathbf{r}}) = -k/a & (\text{at } r=a). \end{cases}$$

   The spherical symmetry again tells us that $\mathbf{E}$ is radial. Thus $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q_{\text{enc}}}{r^2}\,\hat{\mathbf{r}}$ in all three regions. For a gaussian surface with $r < a$ there is no charge enclosed, so $\mathbf{E}(r < a) = 0$. For $a < r < b$ the charge enclosed is $\left(\frac{-k}{a}\right)4\pi a^2 + \int_a^r \left(\frac{-k}{r'^2}\right)4\pi r'^2\,dr' = -4\pi ka - 4\pi k(r-a) = -4\pi kr$, so $\mathbf{E}(a < r < b) = -(k/\epsilon_0 r)\,\hat{\mathbf{r}}$. Finally, for $r > b$ the charge inclosed is the same as in the previous calculation (with $r = b$) plus the surface charge at $r = b$. So $Q_{\text{enc}} = -4\pi kb + 4\pi b^2(k/b) = 0$, thus $\mathbf{E}(r > b) = 0$.

   (b) The spherical symmetry tells us $\mathbf{D}$ must be radial, so $\int \mathbf{D}\cdot d\mathbf{a} = 4\pi r^2 D(r)$ at some radius r. But since there is no free charge anywhere, we must have $bfd = 0$ everywhere. Since $\epsilon_0\mathbf{E} = \mathbf{D} - \mathbf{P} = -\mathbf{P}$, since $\mathbf{P} = 0$ both inside and outside the shell, $\mathbf{E} = 0$ both outside and inside the shell. Within the shell, $\mathbf{E} = -P/\epsilon_0 = -(k/\epsilon_0 r)\,\hat{\mathbf{r}}$. This agrees with part (a) and was far quicker.

7. **Griffiths 4.16** We want to find $\mathbf{D}$ and $\mathbf{E}$ inside the cavity. This is easiest to do by considering the superposition of the original piece of polarized dielectric without a hole and a piece of dielectric in the shape of the cavity possessing opposite polarization. The the fields at the center of the cavity will be the sum of the fields due to the original dielectric (namely $\mathbf{E}_0$ and $\mathbf{D}_0$) with the fields at the center of uniformly polarized objects in the shape of the cavity, which we can call $\mathbf{E}'$ and $\mathbf{D}'$. It is the latter fields that we must determine.

   (a) The fields at the center of a uniformly polarized sphere were found in example 4.3. If $\mathbf{P}$ is the polarization of the original dielectric with a cavity, then $-\mathbf{P}$ is the polarization of the cavity-shaped piece we are superimposing. So $\mathbf{E}' = -\frac{1}{3\epsilon_0}(-\mathbf{P})$. Thus the polarization in the cavity is $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}' = \mathbf{E}_0 + \frac{1}{3\epsilon_0}\mathbf{P}$. Also, since the polarization in the cavity is zero, we have $\mathbf{D} = \epsilon_0\mathbf{E} = \epsilon_0\mathbf{E}_0 + \frac{1}{3}\mathbf{P} = \mathbf{D}_0 - \mathbf{P} + \frac{1}{3}\mathbf{P} = \mathbf{D}_0 - \frac{2}{3}\mathbf{P}$.

   (b) A long thin needle with polarization $-\mathbf{P}$ looks like a bunch of dipoles sitting end to end in a long line, like Figure 4.11 in the text. Thus the net charge that contributes to the electric field at the center of the needle are positive and negative charges on the ends of the needle. But for a very long and very thin needle these will be small charges and far away, so will have negligible contribution. Thus $\mathbf{E}' = 0$, so $\mathbf{E} = \mathbf{E}_0$. Again, in the cavity there is no polarization so $\mathbf{D} = \epsilon_0\mathbf{E} = \epsilon_0\mathbf{E}_0 = \mathbf{D}_0 - \mathbf{P}$.

   (c) The thin wafer shape has the field of a parallel plate capacitor with charge $\sigma_b = \mathbf{P}'\cdot\hat{\mathbf{n}} = -P$ on the upper plate and the opposite charge on the bottom plate. The electric field2 between the plates is then pointing up, in the same direction as $\mathbf{P}$, and has magnitude P/ϵ_0. Thus $\mathbf{E}' = \frac{1}{\epsilon_0}\mathbf{P}$, so $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}' = \mathbf{E}_0 + \frac{1}{\epsilon_0}$. Finally, $\mathbf{D} = \epsilon_0\mathbf{E} = \epsilon_0\mathbf{E}_0 + \mathbf{P} = \mathbf{D}_0$.

8. **Griffiths 4.18** Choose coordinates so that the capacitor is in the xy-plane and $\hat{\mathbf{z}}$ points "up" from the negatively charged plate towards the positively charged plate. Let's start this problem by thinking physically about what will happen. There is free charge placed on the top and bottom plates, which will produce some electric field pointing down. (We will assume the capacitor is big enough in the xy-directions so that the electric field will be only in the z-direction.) But that electric field will polarize the two dielectrics, producing $\mathbf{P}$ pointing in the same direction as $\mathbf{E}$, which in turn induces positive bound surface charge on the bottoms of each dielectric surface and negative bound charge on the top of each dielectric surface. These collections of bound charge will also produce their own electric field, which we also need to take into account. Now let's work through the details.

   (a) The $\mathbf{D}$ field depends only on the free charge, so with $+\sigma$ on the top plate and $-\sigma$ on the bottom plate, the $\mathbf{D}$ field in between the plates will be $\mathbf{D} = \sigma(-\hat{\mathbf{z}})$, which is the $\mathbf{D}$ field between two infinite planes with free surface charges $\pm\sigma$. It has the same value in each of the slabs.

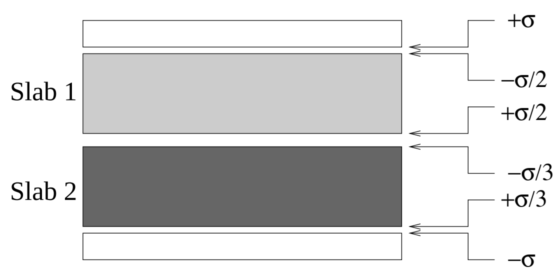
(b) Since we're dealing with linear dielectrics, $\mathbf{D} = \epsilon_0 \epsilon_r \mathbf{E}$. However, ϵ_r is different in the two slabs. Thus in slab 1, $\mathbf{E}_1 = \mathbf{D}/\epsilon_0 \epsilon_r^{(1)} = -\sigma/2\epsilon_0\,\hat{\mathbf{z}}$, and in slab 2, $\mathbf{E}_2 = \mathbf{D}/\epsilon_0 \epsilon_r^{(2)} = -\sigma/1.5\epsilon_0\,\hat{\mathbf{z}} = -2\sigma/3\epsilon_0\,\hat{\mathbf{z}}$.

(c) In a linear dielectric, $\mathbf{P} = \epsilon_0(\epsilon_r - 1)\mathbf{E}$, so in slab 1 we have $\mathbf{P}_1 = \epsilon_0(2-1)\mathbf{E}_1 = -\sigma/2\,\hat{\mathbf{z}}$ and in slab 2 we have $\mathbf{P}_2 = \epsilon_0(1.5-1)\mathbf{E}_2 = -\sigma/3\,\hat{\mathbf{z}}$.

(d) We find the potential difference by integrating $\mathbf{E} \cdot d\ell$ between the two plates. Since $\mathbf{E}$ is uniform in each of the slabs, and points straight down, we get $V = E_1 a + E_2 a = 7a\sigma/6\epsilon_0$.

(e) $\rho_b = -\boldsymbol{\nabla} \cdot \mathbf{P} = 0$ in both slabs. $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$, so remembering that $\hat{\mathbf{n}}$ always points out of each slab, $\sigma_b^{(1)} = +\sigma/2$ at the bottom of slab 1 and minus that at the top of slab 1. $\sigma_b^{(2)} = +\sigma/3$ at the bottom of slab 2 and minus that at the top. See the figure below.

(f) Using all of the charges, we want to recalculate the electric field in each slab. All of the charges are surface charges distributed on (approximately) infinite planes of charge, so we use the result that the electric field due to a single plane of surface charge doesn't depend on the distance from the plane. Inside slab 1 it is as if there was a single plane on top with net surface charge equal to the free charge on the top capacitor plate plus the bound charge on the top of slab 1, namely $\sigma - \sigma/2 = \sigma/2$; and also a single plane below with net surface charge dues to the bound charge on the bottom of slab 1, the top of slab 2, and the bottom of slab 2, and the free charge on the bottom capacitor plate, namely $\sigma/2 - \sigma/3 + \sigma/3 - \sigma = -\sigma/2$. So inside slab 1 the electric field is the same as between two infinite plates with surface charge $\pm\sigma/2$, so $E_1 = \sigma/2\epsilon_0$ (pointing down). Similarly, inside slab 2 there is net surface charge $\sigma - \sigma/2 + \sigma/2 - \sigma/3 = 2\sigma/3$ above and $\sigma/3 - \sigma = -2\sigma/3$ below. So the electric field in slab 2 is $E_2 = 2\sigma/3\epsilon_0$, again pointing down.



Problem 8. Griffiths 3.18

University of California, Berkeley
Physics 110A Fall 2001 Section 1 (*Strovink*)

Problem Set 5

1. Griffiths 5.1

2. Griffiths 5.7

3. Griffiths 5.8

4. Griffiths 5.11

5. Griffiths 5.12

6. Griffiths 5.17

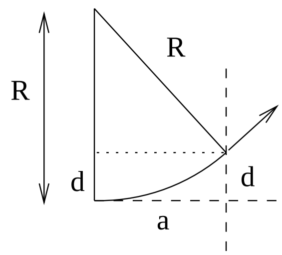
7. Griffiths 5.20

8. Griffiths 5.21

University of California, Berkeley
Physics 110A Fall 2001 Section 1 (*Strovink*)

Solution Set 5 (compiled by Daniel Larson)

1. **Griffiths 5.1** Since the field is pointing into the page, a positive charge would feel a force in the direction $\mathbf{v} \times \mathbf{B}$, which is up. So the charge is **positive**. From example 5.1 we know that momentum is $p = QBR$ where R is the radius of the circle traced out by the charge. Using the pythagorean theorem, and in the figure below, we find $(R-d)^2 + a^2 = R^2 \;\Rightarrow\; R^2 - 2Rd + d^2 + a^2 = R^2 \;\Rightarrow\; R = (a^2+d^2)/2d$. Thus $p = QB\frac{a^2+d^2}{2d}$.



Problem 1. Griffiths 5.1

2. **Griffiths 5.7** First lets calculate the time derivative of the total dipole moment. Recall the definition of $\mathbf{p} = \int_{\mathcal{V}} \rho\mathbf{r}\, d\tau$.
$$\frac{d\mathbf{p}}{dt} = \frac{d}{dt}\int_{\mathcal{V}} \rho\mathbf{r}\, d\tau = \int_{\mathcal{V}} \left(\frac{\partial\rho}{\partial t}\right)\mathbf{r}\, d\tau = -\int_{\mathcal{V}} (\boldsymbol{\nabla}\cdot\mathbf{J})\mathbf{r}\, d\tau,$$
where in the last equality we've used the continuity equation. Now we need to use the hint, so calculate $\boldsymbol{\nabla}\cdot(x\mathbf{J}) = x(\boldsymbol{\nabla}\cdot\mathbf{J}) + \mathbf{J}\cdot(\boldsymbol{\nabla}x) = x(\boldsymbol{\nabla}\cdot\mathbf{J}) + J_x$ because $\boldsymbol{\nabla}x = \hat{\mathbf{x}}$. We are assuming that the current is completely within the volume $\mathcal{V}$, so that means there can be no current leaving through the surface S. But that is true only if $\mathbf{J}\cdot d\mathbf{a} = 0$ on S.
$$\int_{\mathcal{V}} x(\boldsymbol{\nabla}\cdot\mathbf{J})\, d\tau + \int_{\mathcal{V}} J_x\, d\tau = \int_{\mathcal{V}} \boldsymbol{\nabla}\cdot(x\mathbf{J})\, d\tau = \int_S x\mathbf{J}\cdot d\mathbf{a} = 0 \quad\Rightarrow\quad \int_{\mathcal{V}} J_x\, d\tau = -\int_{\mathcal{V}} (\boldsymbol{\nabla}\cdot\mathbf{J})x\, d\tau.$$
We can make the same argument with x replaced by y or z. Putting the three results together gives the vector equation $-\int_{\mathcal{V}}(\boldsymbol{\nabla}\cdot\mathbf{J})\mathbf{r}\, d\tau = \int_{\mathcal{V}} \mathbf{J}\, d\tau$. Combining this with the previous computation, we find $\frac{d\mathbf{p}}{dt} = \int_{\mathcal{V}} \mathbf{J}\, d\tau$.

3. **Griffiths 5.8**

   (a) We can use the intermediate result from example 5.5, namely equation (5.35). In this case we have $s = R$ and $-\theta_1 = \theta_2 = 45°$. We also have four such contributions, one from each side of the square. So $B = 4\frac{\mu_0 I}{4\pi R}\left(\frac{\sqrt{2}}{2} - \frac{-\sqrt{2}}{2}\right) = \mu_0 I\sqrt{2}/\pi R$.

   (b) Generalizing the previous result, $s = R$, $-\theta_1 = \theta_2 = \pi/n$, so $B = n\frac{\mu_0 I}{4\pi R}[\sin(\pi/n) - \sin(-\pi/n)] = \frac{n\mu_0 I}{2\pi R}\sin(\pi/n)$.

   (c) Now taking $n \to \infty$, for small x, $\sin x \approx x$, so for large n we have $\frac{\sin(\pi/n)}{1/n} \approx \frac{\pi/n}{1/n} = \pi$. Thus $B = \mu_0 I/4R$, which is the result in equation (5.38) with $z = 0$.

4. **Griffiths 5.11** We imagine the solenoid to be a series of n circular coils per unit length, each contributing a field $B = \frac{\mu_0 I}{2}\frac{a^2}{(a^2+z^2)^{3/2}}$ to to point at P, where z is the disance along the solenoid's axis between P and the center of the coil (Equation 5.38). To add up the contributions from all the rings, we note that the amount of current flowing in a section of width dz is $nIdz$, so we need to integrate over $z = a\cot\theta$ from one end of the solenoid to the other.
$$B = \int \frac{\mu_0 Ina^2\, dz}{2(a^2+z^2)^{3/2}} = \frac{\mu_0 In}{2}\int_{\theta_1}^{\theta_2} \frac{a^2}{a^3(1+\cot^2\theta)^{3/2}}\left(\frac{-a\, d\theta}{\sin^2\theta}\right) = -\frac{\mu_0 nI}{2}\int_{\theta_1}^{\theta_2} \sin\theta\, d\theta = \frac{\mu_0 nI}{2}(\cos\theta_2 - \cos\theta_1)$$

For an infinite solenoid, $\theta_2 = 0$ and $\theta_1 = \pi$, so $\cos\theta_2 - \cos\theta_1 = 1 - (-1) = 2$. Hence $B = \mu_0 n I$.

5. **Griffiths 5.12** Using equation (5.37), the magnetic force of attraction per unit length between two wires carrying currents I_1 and I_2 is $f_m = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{d}$. Since the current in each wire is $I = \lambda v$, we have $f_m = \frac{\mu_0}{2\pi}\frac{\lambda^2 v^2}{d}$. The electric field of one wire at a distance d is $E = \frac{\lambda}{2\pi\epsilon_0 d}$, so the electric repulsion per unit length is $f_e = \lambda E = \frac{\lambda^2}{2\pi\epsilon_0 d}$. The forces will balance when $f_m = f_e \Rightarrow \mu_0 v^2 = 1/\epsilon_0 \Rightarrow v^2 = 1/\epsilon_0\mu_0$. But recall that these fundamental constants are related to the speed of light: $\mu_0\epsilon_0 c^2 = 1$. Thus the forces will balance when $v = c = 3 \times 10^8$ m/s. Obviously one could never accelerate any physical wires to the speed of light; thus the electric repulsion always dominates.

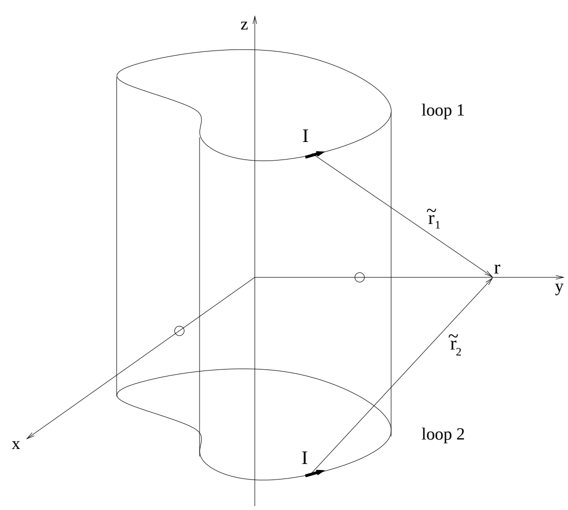
6. **Griffiths 5.17** Let's choose coordinates so that the z-axis runs along the axis of the solenoid. We want to find the magnetic field at any arbitrary point. But we can choose coordinates so that this point is on the y-axis: $\mathbf{r} = (0, y, 0)$. Now we want to look at contributions to the magnetic field from small pieces of current loops, one above and one below $\mathbf{r}$, as shown in the figure. First consider the contribution from a section on loop 1 at position (x', y', z'), located above $\mathbf{r}$. $d\mathbf{l'} = dx'\,\hat{\mathbf{x}} + dy'\,\hat{\mathbf{y}}$. Also, the vector pointing from (x', y', z') to $\mathbf{r} = (0, y, 0)$ is: $\tilde{\mathbf{r}} = -x'\,\hat{\mathbf{x}} + (y - y')\,\hat{\mathbf{y}} - z'\,\hat{\mathbf{z}}$. Thus $d\mathbf{l'} \times \tilde{\mathbf{r}} = (-z'\,dy')\,\hat{\mathbf{x}} + (z'\,dx')\,\hat{\mathbf{y}} + [(y - y')\,dx' + x'\,dy']\,\hat{\mathbf{z}}$. So the contribution to the magnetic field from this piece of loop 1 is:

$$d\mathbf{B}_1 = \frac{\mu_0 I}{4\pi}\frac{d\mathbf{l'} \times \tilde{\mathbf{r}}}{\tilde{r}^3} = \frac{\mu_0 I}{4\pi}\frac{(-z'\,dy')\,\hat{\mathbf{x}} + (z'\,dx')\,\hat{\mathbf{y}} + [(y - y')\,dx' + x'\,dy']\,\hat{\mathbf{z}}}{[(x')^2 + (y - y')^2 + (z')^2]^{3/2}}$$

Now we want to consider the contribution from a section of a coil that is at the same position but below $\mathbf{r}$, i.e. at $(x', y', -z')$. The only difference is that z' changes sign, so the contribution to the magnetic field will be

$$d\mathbf{B}_2 = \frac{\mu_0 I}{4\pi}\frac{d\mathbf{l'} \times \tilde{\mathbf{r}}_2}{\tilde{r}_2^3} = \frac{\mu_0 I}{4\pi}\frac{(z'\,dy')\,\hat{\mathbf{x}} + (-z'\,dx')\,\hat{\mathbf{y}} + [(y - y')\,dx' + x'\,dy']\,\hat{\mathbf{z}}}{[(x')^2 + (y - y')^2 + (-z')^2]^{3/2}}$$

When these two contributions are added, the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ components exactly cancel, leaving only a z-component. Because we have an infinite solenoid, every piece of current above $\mathbf{r}$ has a corresponding piece below, so all x and y components will cancel, and the total magnetic field will point in the z direction. Since we never assumed that $\mathbf{r}$ was either inside or outside the solenoid, this result holds in both cases. Finally, we can use Ampere's law just like in example 5.9 to conclude $\mathbf{B} = 0$ outside the solenoid and $\mathbf{B} = \mu_0 n I\,\hat{\mathbf{z}}$ inside.



Problem 6. Griffiths 5.17

For the toroid, $N/2\pi s \approx n$ as long as the radius of the whole toroid is very large compared to the "radius" of the cross-sectional area. This means that s is about the same at the inner and outer edges of the toroid; in other words, that the coils are not much closer to each other on the inside edge than on the outer edge. If this is the case, then equation (5.58) gives $B = \mu_0 n I$ just like for a straight solenoid.

7. **Griffiths 5.20** Ampere's law says $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$. Taking the divergence of both sides we get $\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 \nabla \cdot \mathbf{J} = -\mu_0 \frac{\partial \rho}{\partial t}$, after using the continuity equation. This is inconsistent with the fact that the divergence of a curl is *always* zero, *unless* we have $\frac{\partial \rho}{\partial t} = 0$, which means we are in the magnetostatic regime. Thus outside of magnetostatics we need to have something else on the right hand side for Ampere's law to be valid; later we'll find out we have to add $\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$. The other Maxwell equations are fine: $\nabla \times \mathbf{E} = 0 \Rightarrow \nabla \cdot (\nabla \times \mathbf{E}) = 0$ is consistent, and there aren't any vanishing second derivatives we can make acting on a divergence.

8. **Griffiths 5.21** At this stage we've just learned about electro- and magnetostatics, so we can consider Maxwell's equations without the time derivatives. Gauss's Law and Ampere's Law would probably stay the same. In analogy with Gauss's law, the divergence of $\mathbf{B}$ would be given by magnetic charges, ρ_m. Let the constant be α_0. Then $\nabla \cdot \mathbf{B} = \alpha_0 \rho_m$. This leads to an analog of Coulomb's law, $\mathbf{F} = \frac{\alpha_0}{4\pi} \frac{q_{m_1} q_{m_2}}{r^2} \hat{\mathbf{r}}$. So by defining a unit of magnetic charge we could measure the force between unit charges at a given distance in order to determine α_0. The moving magnetic charges would presumably create electric fields, in analogy with Ampere's law, so $\nabla \times \mathbf{E} = \beta_0 \mathbf{J}_m$, where β_0 is the constant we would have to measure and $\mathbf{J}_m$ is the magnetic current density. We could determine β_0 by measuring the force between two wires carrying a specified amount of magnetic current. If magnetic charge is conserved, the there should be a corresponding continuity equation: $\nabla \cdot \mathbf{J}_m = -\frac{\partial \rho_m}{\partial t}$.

   To get the force law, the first guess for the force on a magnetic charge q_m could be $q_m[\mathbf{B} + (\mathbf{v} \times \mathbf{E})]$. However, the dimensions are wrong, because E has the same units as vB. So we need to divide the second part, $(\mathbf{v} \times \mathbf{E})$ by something with dimensions of velocity-squared. The obvious choice is the speed of light, especially in light of the relationship $\mu_0 \epsilon_0 c^2 = 1$. So the total force law would be:

$$\mathbf{F} = q_e[\mathbf{E} + (\mathbf{v} \times \mathbf{B})] + q_m\left[\mathbf{B} - \frac{1}{c^2}(\mathbf{v} \times \mathbf{E})\right].$$

   (The minus sign is to keep consistent with special relativity. For more discussion of magnetic charge in terms of the full Maxwell equations, you could look ahead to Section 7.3.4 in the text.)

University of California, Berkeley
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Problem Set 6

1. Griffiths 5.24

2. Griffiths 5.25

3. Griffiths 5.39

4. Griffiths 5.41

5. Griffiths 5.56

6. Griffiths 6.10

7. Griffiths 6.12

8. Griffiths 6.13

University of California, Berkeley
Physics 110A Fall 2001 Section 1 (*Strovink*)

Solution Set 6 (compiled by Daniel Larson)

1. **Griffiths 5.24** If $\mathbf{B}$ is uniform, then it is not a function of position, so any derivative of it vanish. In particular, $\nabla\times\mathbf{B}=0$ and $\nabla\cdot\mathbf{B}=0$. One can also check using cartesian coordinates that $\nabla\times\mathbf{r}=0$ and $\nabla\cdot\mathbf{r}=3$. Using these results we find $\nabla\cdot\mathbf{A}=-\frac{1}{2}\nabla\cdot(\mathbf{r}\times\mathbf{B})=-\frac{1}{2}[\mathbf{B}\cdot(\nabla\times\mathbf{r})-\mathbf{r}\cdot(\nabla\times\mathbf{B})]=0$. Also, using the fact that $(\mathbf{r}\cdot\nabla)\mathbf{B}=0$ since $\mathbf{B}$ is uniform, $\nabla\times\mathbf{A}=-\frac{1}{2}\nabla\times(\mathbf{r}\times\mathbf{B})=-\frac{1}{2}[(\mathbf{B}\cdot\nabla)\mathbf{r}-(\mathbf{r}\cdot\nabla)\mathbf{B}+\mathbf{r}(\nabla\cdot\mathbf{B})-\mathbf{B}(\nabla\cdot\mathbf{r})]=-\frac{1}{2}[(\mathbf{B}\cdot)\mathbf{r}-3\mathbf{B}]$. Now, $(\mathbf{B}\cdot\nabla)\mathbf{r}=\left(B_x\frac{\partial}{\partial x}+B_y\frac{\partial}{\partial y}+B_z\frac{\partial}{\partial z}\right)(x\,\hat{\mathbf{x}}+y\,\hat{\mathbf{y}}+z\,\hat{\mathbf{z}})=B_x\,\hat{\mathbf{x}}+B_y\,\hat{\mathbf{y}}+B_z\,\hat{\mathbf{z}}=\mathbf{B}$. Thus $\nabla\times\mathbf{A}=-\frac{1}{2}(\mathbf{B}-3\mathbf{B})=\mathbf{B}$. We can add any constant to bfa without changing the divergence and curl, so the result is unique up to the addition of a constant vector field.

2. **Griffiths 5.25**

   (a) Let's assume that $\mathbf{A}$ points in the same direction as the current, namely the $\hat{\mathbf{z}}$ direction. Furthermore, the vector potential should be independent of ϕ and z because the infinite wire is symmetric with respect to translations and rotations about the z-axis. So we make the guess that $\mathbf{A}=A(s)\,\hat{\mathbf{z}}$. Using the formulas for taking divergence and curl in cylindrical coordinates, we find $\nabla\cdot\mathbf{A}=\frac{\partial}{\partial z}A(s)=0$ and $\nabla\times\mathbf{A}=-\frac{\partial}{\partial s}A(s)\,\hat{\boldsymbol{\phi}}$. Since $\mathbf{B}=\frac{\mu_0 I}{2\pi s}\,\hat{\boldsymbol{\phi}}$, we must have $\frac{\partial A}{\partial s}=-\frac{\mu_0 I}{2\pi s}\Rightarrow A(s)=-\frac{\mu_0 I}{2\pi}\ln s$. For the units to make sense, we need an arbitrary length in the logarithm, so finally $\mathbf{A}=-\frac{\mu_0 I}{2\pi}\ln(s/a)\,\hat{\mathbf{z}}$. (Note that putting "$a$" in the log is the same as adding a constant, so it doesn't change the divergence or curl of $\mathbf{A}$.)

   (b) First we need to find the magnetic field inside the wire, for $s<R$. Ampere's law gives $\oint\mathbf{B}\cdot d\mathbf{l}=2\pi sB(s)=\mu_0 I_{\text{enc}}=\mu_0 I\frac{\pi s^2}{\pi R^2}\Rightarrow\mathbf{B}=\frac{\mu_0 I s}{2\pi R^2}\,\hat{\boldsymbol{\phi}}$. We assume that $\mathbf{A}$ is of the same form as in part a, so $\frac{\partial A}{\partial s}=-\frac{\mu_0 I s}{2\pi R^2}\Rightarrow\mathbf{A}=-\frac{\mu_0 I}{4\pi R^2}(s^2-b^2)\,\hat{\mathbf{z}}$ where b is the constant of integration. For $s>R$ the B-field and thus $\mathbf{A}$ look the same as in part (a), except that we need $\mathbf{A}$ to be continuous at $s=R$. We can accomplish this by taking $a=b=R$. So finally, $\mathbf{A}=\begin{cases}-\frac{\mu_0 I}{4\pi R^2}(s^2-R^2)\,\hat{\mathbf{z}}, & \text{for } s\le R;\\ -\frac{\mu_0 I}{2\pi}\ln(s/R)\,\hat{\mathbf{z}}, & \text{for } s\ge R.\end{cases}$

3. **Griffiths 5.39**

   (a) Using the right-hand-rule, positive charges will be deflected down.

   (b) Charge accumulates on the bottom and top plates until the electric force balances the magnetic force. For a single charge, this means $qE=qvB\Rightarrow E=vB$. The field between two large, charged plates is essentially uniform, hence $V=Et$. So $V=vBt$. The bottom is at a higher potential, because that is where the positive charge is.

   (c) A current flowing to the right can be considered as positive charges flowing right or negative charges flowing left. If negative charges flow left, the will also feel a magnetic force downward, and thus *negative* charges will build up on the bottom plate. The potential difference between the top and bottom will be the same, but this time the top plate will be at higher potential.

4. **Griffiths 5.41** In cylindrical coordinates $\mathbf{B}$ is in the $\hat{\mathbf{z}}$ direction (either into or out of the page) and depends only on the radial distance s. The particle traveling in the shaded region is assumed to be in the $x-y$ plane at a location specified by the coordinate $\mathbf{r}$, with tangent vector $d\mathbf{l}=dr\,\hat{\mathbf{r}}+rd\phi\,\hat{\boldsymbol{\phi}}$. If the particle starts from the origin, it cannot have any angular momentum relative to the origin. If it emerges from the shaded region on a radial trajectory, its angular momentum is $\mathbf{r}\times\mathbf{p}=0$. So if we can show that the particle acquires no angular momentum throughout its motion, we will have proven that it must emerge on a radial trajectory. We also know that $\int\mathbf{B}\cdot d\mathbf{a}=\int\mathbf{B}2\pi r\,dr=0$. Recall that the torque about the origin is $\mathbf{N}=\frac{d\mathbf{L}}{dt}=\mathbf{r}\times\mathbf{F}$.

$$\mathbf{L}=\int\frac{d\mathbf{L}}{dt}\,dt=\int(\mathbf{r}\times\mathbf{F})\,dt=\int\mathbf{r}\times q(\mathbf{v}\times\mathbf{B})\,dt=q\int\mathbf{r}\times(d\mathbf{l}\times\mathbf{B})=q\left[\int(\mathbf{r}\cdot\mathbf{B})\,d\mathbf{l}-\int\mathbf{B}(\mathbf{r}\cdot d\mathbf{l})\right],$$

where we have used $\mathbf{v}dt = d\mathbf{l}$ and the BAC-CAB rule for a triple cross product. Now, since the particle is in the xy-plane and $\mathbf{B}$ is normal to the page, $\mathbf{r}\cdot\mathbf{B} = 0$. Also, $\mathbf{r}\cdot d\mathbf{l} = r\,\hat{\mathbf{r}}\cdot(dr\,\hat{\mathbf{r}} + rd\phi\,\hat{\boldsymbol{\phi}}) = r\,dr$. So $\mathbf{L} = -\frac{q}{2\pi}\int \mathbf{B}2\pi r\,dr = 0$ because $B_x = B_y = 0$ and $\int B_z 2\pi r\,dr = 0$ by assumption. Thus the particle emerges with zero total angular momentum, which means it must be traveling along a radial line.

5. **Griffiths 5.56**

   (a) The angular momentum of a ring is $\mathbf{L} = I\omega\,\hat{\mathbf{z}}$ with $I = MR^2$, and its dipole moment will be $\mathbf{m} = IA\,\hat{\mathbf{z}} = \frac{Q}{2\pi/\omega}\pi R^2\,\hat{\mathbf{z}} = \frac{1}{2}Q\omega R^2\,\hat{\mathbf{z}}$. Thus $\mathbf{m} = \frac{Q}{2M}\mathbf{L}$. So the gyromagnetic ration is $g = \frac{Q}{2M}$.

   (b) Because g is independent of the radius, the same applies to all infinitesimal rings of charge. We could calculate the total angular momentum of a spinning sphere by adding up the contributions from each ring, just as we could get the total magnetic moment by adding up the contributions from each ring. Since each ring will contribute to the magnetic moment and angular momentum in the same proportion, the ratio of total dipole moment to angular momentum will be the same as in part (a), $g = \frac{Q}{2M}$.

   (c) If the electron has angular momentum $\frac{1}{2}\hbar$ then the dipole moment m will be

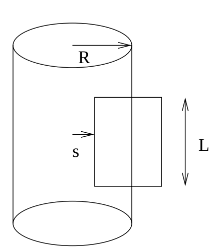
$$m = \frac{e}{2m_e}\frac{1}{2}\hbar = \frac{e\hbar}{4m_e} = \frac{(1.60\times 10^{-19}\text{ C})(1.05\times 10^{-34}\text{ Js})}{4(9.11\times 10^{-31}\text{ kg})} = 4.61\times 10^{-24}\text{ A m}^2.$$

6. **Griffiths 6.10** Because the magnetization is uniform, $\nabla\times\mathbf{M} = 0$, so there is no volume bound current, but only a surface bound current $K_b = M$, wrapping around the rod like the current in a solenoid. For $a \ll L$, a is much smaller than the radius of the toroid, so in equation (5.58), we can treat s as the radius of the toroid. Then $\frac{NI}{2\pi s}$ is the amount of current flowing around the toroid, per unit length, which is exactly what we mean by surface current. Thus the B-field inside a complete, magnetized toroid is $\mathbf{B} = \mu_0\frac{NI}{2\pi s}\,\hat{\boldsymbol{\phi}} = \mu_0 K_b\,\hat{\boldsymbol{\phi}} = \mu_0\mathbf{M}$.

   But part of the toroid is cut out, which we can treat as a bunch of square loops carrying the opposite current; hence they will produce a magnetic field in a direction opposite to the one produced by the rest of the toroid. In problem (5.8) we found the B-field at the center of a square loop: $B = \mu_0 I\sqrt{2}/\pi R$. In this case $R = a/2$ (the perpendicular distance from the center of the loop to its side). We assume that $w \ll a$, so we can think of the gap as a single square loop with all the current running around it. Thus $I = K_b w = Mw$. So the missing piece of the toroid contributes $-2\sqrt{2}\mu_0 Mw/\pi a$. So at the center of the gap, $\mathbf{B} = \mu_0\mathbf{M}\Big(1 - \frac{2\sqrt{2}w}{\pi a}\Big)$.

7. **Griffiths 6.12**

   (a) There is a surface bound current $\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}} = kR\,\hat{\boldsymbol{\phi}}$ and a volume current $\mathbf{J}_b = \nabla\times\mathbf{M} = -k\,\hat{\boldsymbol{\phi}}$. Since all the current is circumferential, we can think of the situation as the superposition of lots of coaxial solenoids of different radii. So immediately we conclude $\mathbf{B} = 0$ outside the cylinder. Now we can draw a square amperian loop that has one side parallel to the z-axis inside the cylinder, and the opposite side parallel to the z-axis outside. We know the B-field should be pointing in the z-direction, so we'll get no contribution to the line integral from the other two sides. Since $B = 0$ outside, the only section of the loop that contributes is the piece inside the cylinder parallel to the z-axis. $\oint\mathbf{B}\cdot d\mathbf{l} = BL = \mu_0 I_{\text{enc}} = \mu_0\big[\int J_b da + K_b L\big] = \mu_0[-kL(R-s) + kRL] = \mu_0 kLs$. ($L(R-s)$ is the area of the amperian loop inside the cylinder.) So $\mathbf{B} = \mu_0 ks\,\hat{\mathbf{z}}$ inside.



Problem 7. Griffiths 6.12

(b) Since $\mathbf{M}$ is the only object in this problem that picks out a direction in space, we know $\mathbf{H}$ must also point in the z-direction. However, using the same amperian loop as in part (a), $\oint \mathbf{H} \cdot d\mathbf{l} = HL = \mu_0 I_{f_{\text{enc}}} = 0$ because there are no free currents. Thus $\mathbf{H} = 0$, so $\mathbf{B} = \mu_0 \mathbf{M}$. Outside, $\mathbf{M} = 0$ so $\mathbf{B} = 0$; inside $\mathbf{M} = ks\,\hat{\mathbf{z}}$, so $\mathbf{B} = \mu_0 ks\,\hat{\mathbf{z}}$.

8. **Griffiths 6.13** We assume that the cavities as small enough so that the fields are essentially uniform inside of them. We treat the cavities by considering the superposition of a piece of material without cavities and small, cavity-shaped objects with opposite magnetization.

(a) The B-field of a uniformly magnetized sphere is $\frac{2}{3}\mu_0\mathbf{M}$, so the contribution to the B-field from the cavity is the same as the contribution from a uniformly magnetized sphere with magnetization $-\mathbf{M}$, namely $\mathbf{B}_{\text{cav}} = -\frac{2}{3}\mu_0\mathbf{M}$. Thus with the sphere removed $\mathbf{B} = \mathbf{B}_0 - \frac{2}{3}\mu_0\mathbf{M}$. Inside the real cavity, $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B}$ because there is no magnetization, so $= \frac{1}{\mu_0}(\mathbf{B_0} - \frac{2}{3}\mu_0\mathbf{M}) = \mathbf{H_0} + \mathbf{M} - \frac{2}{3}\mathbf{M} \Rightarrow \mathbf{H} = \mathbf{H_0} + \frac{1}{3}\mathbf{M}$.

(b) For a long, thin, cylindrical cavity with uniform magnetization $-\mathbf{M}$ there is only surface current $K_b = -M$, which looks like a solenoid. So the B-field at the center is $\mu_0 K_b = -\mu_0 M$. Adding this to the contribution from the cavity-less material, we find $\mathbf{B} = \mathbf{B}_0 - \mu_0\mathbf{M}$. Then $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} = \frac{1}{\mu_0}(\mathbf{B}_0 - \mu_0\mathbf{M}) = \frac{1}{\mu_0}\mathbf{B}_0 - \mathbf{M} \Rightarrow \mathbf{H} = \mathbf{H}_0$.

(c) For the wafer shaped cavity, the bound currents run around the outside edge, so if the wafer has a large radius and is very thin, those currents will be very small and far away from the center and will contribute virtually no magnetic field. Thus $\mathbf{B} = \mathbf{B}_0$. Then $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B}_0 = \mathbf{H}_0 + \mathbf{M} \Rightarrow \mathbf{H} = \mathbf{H}_0 + \mathbf{M}$.

University of California, Berkeley
Physics 110A Fall 2001 Section 1 (*Strovink*)

Problem Set 7

1. Griffiths 6.17

2. Griffiths 6.21

3. Griffiths 6.26

4. Griffiths 7.3

5. Griffiths 7.7

6. Griffiths 7.11

7. Griffiths 7.17

8. Griffiths 7.48

University of California, Berkeley
Physics 110A Fall 2001 Section 1 (*Strovink*)

Solution Set 7 (compiled by Daniel Larson)

1. **Griffiths 6.17** In a linear material, we know H is proportional to B: $\mathbf{B} = \mu\mathbf{H} = \mu_0(1+\chi_m)\mathbf{H}$, so for a long wire it should be circumferential. We can then use Ampere's law to find $\mathbf{H}$ from the free current, and then get B from H. As usual, we draw an amperian loop around the wire:

$$\oint \mathbf{H}\cdot d\mathbf{l} = 2\pi s H(s) = I_{f_{\text{enc}}} = \begin{cases} I(s^2/a^2), & (s < a) \\ I & (s > a) \end{cases}$$

$$H(s) = \begin{cases} \frac{Is}{2\pi a^2}, & (s < a) \\ \frac{I}{2\pi s}, & (s > a) \end{cases} \Rightarrow B(s) = \begin{cases} \frac{\mu_0(1+\chi_m)Is}{2\pi a^2}, & (s < a) \\ \frac{\mu_0 I}{2\pi s}, & (s > a) \end{cases}.$$

In a linear material, $J_b = \chi_m J_f = \chi_m \frac{I}{\pi a^2}$ (using the fact that I is uniform over the area of the wire) and points in the same direction as I. $\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}} = \chi_m\mathbf{H}(a)\times\hat{\mathbf{n}} \Rightarrow K_b = \frac{\chi_m I}{2\pi a}$ in the direction *opposite* from I (using the right-hand-rule). The total bound current is $I_b = \pi a^2 J_b + 2\pi a K_b = \chi_m I - \chi_m I = 0$ as it must be.

2. **Griffiths 6.21**

   (a) We need to compute the work it takes to bring the magnetic dipole in from infinity to the origin and rotate it to its final configuration. First, bring the dipole to the origin along a trajectory in which $\mathbf{m}$ is always perpendicular to $\mathbf{B}$ so that there is no force on the dipole and hence no work done. For simplicity, imagine $\mathbf{B}$ is uniform and points in the $\hat{\mathbf{y}}$ direction. Then we can slide a dipole (pointing in the $\hat{\mathbf{x}}$ direction) in along the x-axis. All the work comes from rotating the dipole in the presence of the B-field. The torque exerted by the B-field is $\mathbf{N} = \mathbf{m}\times\mathbf{B} = mB\sin\theta\,\hat{\mathbf{z}}$ where θ is the angle between $\mathbf{m}$ and $\mathbf{B}$ (initially $\pi/2$); this is opposite the torque we must exert in order to rotate the dipole. So to move the dipole from an angle of $\pi/2$ with respect to $\mathbf{B}$ to some other angle θ we must do an amount of work $U = \int_{\pi/2}^{\theta} mB\sin\theta'\,d\theta' = mB(-\cos\theta')|_{\pi/2}^{\theta} = -mB\cos\theta = -\mathbf{m}\cdot\mathbf{B}$.

   (b) We can put the first diple at the origin. It produces a magnetic field $\mathbf{B}_1 = \frac{\mu_0}{4\pi r^3}[3(\mathbf{m}_1\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{m}_1]$ at any location $\mathbf{r}$. The second dipole, located at $\mathbf{r}$, interacts with this magnetic field as in part (a). Thus $U = -\mathbf{m}_2\cdot\mathbf{B}_1 = -\frac{\mu_0}{4\pi r^3}[3(\mathbf{m}_1\cdot\hat{\mathbf{r}})\mathbf{m}_2\cdot\hat{\mathbf{r}} - \mathbf{m}_2\cdot\mathbf{m}_1] = \frac{\mu_0}{4\pi r^3}[\mathbf{m}_1\cdot\mathbf{m}_2 - 3(\mathbf{m}_1\cdot\hat{\mathbf{r}})(\mathbf{m}_2\cdot\hat{\mathbf{r}})]$.

   (c) From the figure, $\mathbf{m}_i\cdot\hat{\mathbf{r}} = m_i\cos\theta_i$ for $i = 1$ or 2, and $\mathbf{m}_1\cdot\mathbf{m}_2 = m_1 m_2\cos(\theta_1-\theta_2) = m_1 m_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)$. So $U = \frac{\mu_0 m_1 m_2}{4\pi r^3}[\cos(\theta_1-\theta_2) - 3\cos\theta_1\cos\theta_2] = \frac{\mu_0 m_1 m_2}{4\pi r^3}[\sin\theta_1\sin\theta_2 - 2\cos\theta_1\cos\theta_2]$. A stable configuration occurs when the energy is at a minimum.

$$\frac{\partial U}{\partial\theta_1} = \frac{\mu_0 m_1 m_2}{4\pi r^3}(\cos\theta_1\sin\theta_2 + 2\sin\theta_1\cos\theta_2) = 0 \quad\Rightarrow\quad 2\sin\theta_1\cos\theta_2 = -\cos\theta_1\sin\theta_2$$

$$\frac{\partial U}{\partial\theta_2} = \frac{\mu_0 m_1 m_2}{4\pi r^3}(\sin\theta_1\cos\theta_2 + 2\cos\theta_1\sin\theta_2) = 0 \quad\Rightarrow\quad 2\sin\theta_1\cos\theta_2 = -4\cos\theta_1\sin\theta_2$$

   So we need $\cos\theta_1\sin\theta_2 = \sin\theta_1\cos\theta_2 = 0$. This will happen for either $\sin\theta_1 = \sin\theta_2 = 0 \Rightarrow$ (*i*) $\rightarrow\rightarrow$ or (ii) $\rightarrow\leftarrow$; or if $\cos\theta_1 = \cos\theta_2 = 0 \Rightarrow$ (*iii*) $\uparrow\uparrow$ or (iv) $\uparrow\downarrow$. We know that the lowest energy configuration will have $\mathbf{m}$ lined up with $\mathbf{B}$. This only happens in (i) and (iv), so they are the stable minima. To find the absolute minimum, we need to calculate U. For situation (i) we have $\theta_1 = \theta_2 = 0$ so $U = \frac{\mu_0 m_1 m_2}{4\pi r^3}(-2)$ whereas for (iv) we have $\theta_1 = -\theta_2 = \pi/2$, so $U = \frac{\mu_0 m_1 m_2}{4\pi r^3}(-1)$. Thus the most stable configuration is the one with the lowest energy, namely (i) where the magnetic moments are lined up along the line joining them: $\rightarrow\rightarrow$.

   (d) Using the result from part (c), the most stable configuration should be when the dipoles all form one line, pointing in one direction: $\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow$.

3. **Griffiths 6.26** The angle θ_1 is related to the components of B_1 which are parallel and perpendicular to the interface: $\tan\theta_1 = \frac{B_1^{\parallel}}{B_1^{\perp}}$. The same relation holds for θ_2 and B_2. The perpendicular components of B are continuous across the boundary, so $B_1^{\perp} = B_2^{\perp}$. We also know that the parallel components of H are continuous across the boundary, since there is no free surface current. Since $\mathbf{B} = \mu\mathbf{H}$ this gives: $H_1^{\parallel} = H_2^{\parallel} \Rightarrow \frac{1}{\mu_1}B_1^{\parallel} = \frac{1}{mu_2}B_2^{\parallel}$. Putting these together:

$$\frac{\tan\theta_2}{\tan\theta_1} = \frac{B_2^{\parallel}}{B_2^{\perp}}\frac{B_1^{\perp}}{B_1^{\parallel}} = \frac{B_2^{\parallel}}{B_1^{\parallel}} = \frac{\mu_2}{\mu_1}$$

4. **Griffiths 7.3**

   (a) To find the resistance, we need to look at the ration of the potential difference to the current flowing between to metal objects. Any currents flowing will leave conductor 1 and flow to conductor 2. So we can find the current by enclosing conductor 1 with a surface and then evaluating $I = \int \mathbf{J}\cdot d\mathbf{a}$. This equation is exactly what we need. First, Gauss's law tells us $\int \mathbf{E}\cdot d\mathbf{a} = \frac{1}{\epsilon_0}Q_{\text{enc}}$, while Ohm's law gives $\mathbf{J} = \sigma\mathbf{E}$ and $V = IR$. We assume there are no free charges floating around in our conducting material, so Q_{enc} is simply the charge on the first object, which is related to the capacitance of the system by $Q = CV$. These are all the ingredients we need.

$$I = \int \mathbf{J}\cdot d\mathbf{a} = \sigma\int \mathbf{E}\cdot d\mathbf{a} = \frac{\sigma}{\epsilon_0}Q = \frac{\sigma}{\epsilon_0}CV = \frac{\sigma}{\epsilon_0}CIR \;\Rightarrow\; R = \frac{\epsilon_0}{\sigma C}.$$

   (b) We apply a potential difference V_0 between objects 1 and 2 and then allow the charge to leak off. The voltage at any time is given by $V(t) = I(t)R = -\frac{dQ}{dt}R$, where the minus sign comes because we assume the current I is positive, but we know the charge Q is decreasing. We also know that $V = Q/C$, so that tells us $\frac{dV}{dt} = \frac{1}{C}\frac{dQ}{dt}$, because capacitance is just a constant. Thus $V(t) = -RC\frac{dV}{dt} \Rightarrow \frac{dV}{dt} = -\frac{1}{RC}V(t) \Rightarrow V(t) = V(0)e^{-t/RC} = V_0e^{-t/RC}$. Then the time constant $\tau = RC = \epsilon_0/\sigma$.

5. **Griffiths 7.7**

   (a) Current will flow due to the changing flux in the loop formed by the bar and the wire. The total flux through the loop is $\Phi = BA$. If the bar is moving at speed v to the right, the area is changing at a rate of $\frac{dA}{dt} = lv$. Thus $\mathcal{E} = -\frac{d\Phi}{dt} = -Blv$. Then $\mathcal{E} = IR \Rightarrow I = Blv/R$. The minus sign just refers to the direction, but it is easier to figure that out using Lenz's law. Since the flux into the page is increasing, the current will flow to produce flux coming out of the page, so the current will be going *down* through the resistor.

   (b) There is magnetic force on the bar because there is a current flowing in the presence of a magnetic field. $F = \int I d\mathbf{l}\times\mathbf{B} = IlB = B^2l^2v/R$ and it points to the left, which is the direction of $d\mathbf{l}\times\mathbf{B}$.

   (c) The force on the bar is slowing it down so we take it to be negative.

$$F = -\frac{1}{R}B^2l^2v = ma = m\frac{dv}{dt} \;\Rightarrow\; \frac{dv}{dt} = -\frac{B^2l^2}{Rm}\,v \;\Rightarrow\; v(t) = v_0e^{-B^2l^2t/Rm}.$$

   (d) The energy goes into heading the resisitor. The power delivered to the resisitor is

$$P = \frac{dW}{dt} = I^2R = \frac{B^2l^2}{R}v_0^2e^{-2\alpha t},\text{ where } \alpha = \frac{B^2l^2}{Rm};\;\Rightarrow\; \frac{dW}{dt} = \alpha mv_0^2e^{-2\alpha t}.$$

   The bar keeps slowing down, but takes an infinite amount of time to stop. During this time, the total energy delivered to the resistor is

$$W = \alpha mv_0^2\int_0^\infty e^{-2\alpha t}\,dt = \alpha mv_0^2\left.\frac{e^{-2\alpha t}}{-2\alpha}\right|_0^\infty = \alpha mv_0^2\frac{1}{2\alpha} = \frac{1}{2}mv_0^2.$$

6. **Griffiths 7.11** Let l be the width of the loop, and s be the distance between the top edge of the loop and the bottom of the region of B-field. The flux through the loop is $\Phi = Bla$, so $\mathcal{E} = -\frac{d\Phi}{dt} = -Bl\frac{ds}{dt}$. Let's only consider magnitudes and drop the minus sign. Since $\frac{ds}{dt} = v(t)$, the velocity of the loop at time t, we have $\mathcal{E} = Blv = IR$, assuming the loop has resistance R. Then $I = Blv/R$ is the current flowing in the loop. As the loop falls, the flux into the page is decreasing, so the current flows in a clockwise direction to oppose the change in flux. But the part of the loop still in the region of magnetic field will feel a force because there is a current in a magnetic field. The forces on the two sides will cancel, leaving an upward force of magnitude $F = IlB = B^2l^2v/R$. This force opposes the force of gravity, $F_g = mg$ which pulls the loop downward. The loop will have reached terminal velocity, v_t, when these two forces balance: $mg = B^2l^2v_t/R \;\Rightarrow\; v_t = (mgR)/(B^2l^2)$.

   To find the velocity as a function of time, we need Newton's second law: $F_{\text{net}} = ma = m\frac{dv}{dt} = mg - \frac{B^2l^2}{R}v$ where I have taken the downward direction to be positive. Letting $\alpha = B^2l^2/mR$, we have $v_t = g/\alpha$, and we get a differential equation for the velocity:

$$\frac{dv}{dt} = g - \alpha v \;\Rightarrow\; \frac{dv}{g - \alpha v} = dt \;\Rightarrow\; -\frac{1}{\alpha}\ln(g - \alpha v) = t + \text{const.} \;\Rightarrow\; g - \alpha v = Ae^{-\alpha t}$$

   Since the loop starts at rest at $t = 0$, the constant $A = g$. Thus $v(t) = \frac{g}{\alpha}(1 - e^{-\alpha t}) = v_t(1 - e^{-\alpha t})$. At 90% of terminal velocity we have $v/v_t = 0.9 = 1 - e^{-\alpha t} \;\Rightarrow\; e^{-\alpha t} = 0.1 \;\Rightarrow\; t = \frac{1}{\alpha}\ln 10 = \frac{v_t}{g}\ln 10$.

   To get a numerical answer, we need various properties of aluminum and the dimensions of the loop. Assume the loop is square, with sides l and cross-sectional area A. The resistivity is $\rho = \frac{1}{\sigma} = 2.65 \times 10^{-8}$ Ω m; the mass density is $\eta = 2.7 \times 10^3$ kg/m^3; $g = 9.8$ m/s^2; and $B = 1$ T. The resistance of a piece of metal with uniform cross-sectional area A and length L is $R = \frac{L}{A\sigma}$, so in this case we have $R = \frac{4l\rho}{A}$.

$$v_t = \frac{mgR}{B^2l^2} = \frac{(\eta A4l)g(4l\rho/A)}{B^2l^2} = \frac{16\eta g\rho}{B^2} = 1.1 \text{ cm/s}; \;\Rightarrow\; t_{90\%} = \frac{v_t}{g}\ln 10 = 2.8 \text{ ms}$$

   Finally, if the loop were cut, no current would flow, so there wouldn't be any force to oppose gravity and the loop would fall freely under the force of gravity.

7. **Griffiths 7.17**

   (a) We assume that the solenoid is relatively long, so the only magnetic field in the loop is the uniform B-field inside the solenoid, namely $B = \mu_0 nI$. Thus the flux passing through the loop is $\Phi = \pi a^2 B = \pi a^2\mu_0 nI \;\Rightarrow\; \mathcal{E} = -\pi a^2\mu_0 n\frac{dI}{dt}$. The negative sign just refers to the direction, which is easier to find using Lenz's law, so we'll ignore it. The magnitude of the current passing through the resistor is given by $\mathcal{E} = I_r R \;\Rightarrow\; I_r = \frac{1}{R}\pi a^2\mu_0 nk$. The flux due to the solenoid is pointing to the right and is increasing, thus the current in the loop will flow in order to produce a flux inside the loop pointing to the left, which is opposite the direction of the current flowing in the solenoid, or to the right in the picture in the text.

   (b) When the solenoid is pulled out and reinserted there is lots of changes going on in the flux, most of them very complicated. But all we need to know to get the total charge is the total change in flux.

$$\Delta Q = \int I\,dt = \int \frac{\mathcal{E}}{R} = \int -\frac{1}{R}\frac{d\Phi}{dt} = -\frac{1}{R}(\Phi_f - \Phi_i) \;\Rightarrow\; \Delta Q = \frac{1}{R}\Delta\Phi \quad \text{(in magnitude)}$$

   Initially there is flux $\Phi_i = \pi a^2\mu_0 nI$ pointing to the right, and at the end there is the same amount of flux pointing in the opposite direction, the net change in flux is $\Delta\Phi = 2\pi a^2\mu_0 nI$, which means $\Delta Q = \frac{1}{R}2\pi a^2\mu_0 nI$.

8. **Griffiths 7.48** Starting with Equation (5.3), we have $qBR = mv$. Keeping R fixed, we can differentiate with respect to time: $qR\frac{dB}{dt} = m\frac{dv}{dt} = ma = F = qE$. Thus $E = R\frac{dB}{dt}$, where B is evaluated at the radius of the electron's orbit, R. From Faraday's law we know $\oint \mathbf{E}\cdot d\mathbf{l} = -\frac{d\Phi}{dt}$, so if we take the loop to be the electron's orbit at radius R, $2\pi RE = -\frac{d\Phi}{dt}$. Combining this with the previous result we can solve for B:

$\frac{dB}{dt} = -\frac{1}{2\pi R^2}\frac{d\Phi}{dt} \Rightarrow B = -\frac{1}{2}\left(\frac{\Phi}{\pi R^2}\right) + C$ where C is some integration constant. If $B = 0$ when $t = 0$, there will be no flux through the loop, so the constant must be zero. But this means $B(R) = -\frac{1}{2}\left(\frac{\Phi}{\pi R^2}\right)$. The term in parentheses is simply the total field throughout the orbit (flux) divided by the area of the orbit, namely the average field. Thus the average field over the orbit is twice the value of the field at the circumference (in magnitude).

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Problem Set 8

1. Griffiths 7.21

2. Griffiths 7.23

3. Griffiths 7.26

4. Griffiths 7.58

5. Griffiths 7.30

6. Griffiths 7.31

7. Griffiths 7.37

8. Griffiths 7.50

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Solution Set 8 (compiled by Daniel Larson)

1. **Griffiths 7.21** The emf is the time-derivative of the flux due to the small loop that passes through the big loop, namely $\mathcal{E} = -\frac{d\Phi}{dt} = -M\frac{dI}{dt} = -Mk$ where we have used the definition of the mutual inductance, that the flux through the big loop is proportional to the current in the small loop, and the proportionality constant is M. So all we need to do is calculate M. However, it is quite difficult to calculate the flux due to the small loop, because the B-field due to a square loop is rather complicated. Instead, we can use the equality of mutual inductances and find M by calculating the flux through the small loop due to a current in the big loop. This is much easier, because the big loop is essentially two long wires, symmetrically placed on either side of the little loop. The field from one long wire is $B = \frac{\mu_0 I}{2\pi s} \Rightarrow \Phi_1 = \frac{\mu_0 I}{2\pi}\int_a^{2a}\frac{1}{s}a\,ds = \frac{\mu_0 I a}{2\pi}\ln 2$. The flux from the two wires are the same, so we multiply the above result by 2 and divide by I to find $M = (\mu_0 a\ln 2)/\pi)$. The magnitude of the emf is then $\mathcal{E} = (\mu_0 ka\ln 2)/\pi$.

   To determine the direction of the current in the big loop we will use Lenz's law, and also the fact that the B-field lines produced by the square loop are all closed curves. That means that every line of flux heads into the page inside the square loop and comes back out of the page somewhere outside the square loop. Since the big loop encloses all of the flux heading into the page through the center of the small square loop, but only some of the flux coming back out of the page, the net flux through the big loop is into the page. When the current in the small loop decreases, the net flux into the page decreases, so a counterclockwise current is induced in the large loop to oppose the change in flux.

2. **Griffiths 7.23** We need to compute the flux passing through the loop due to the current flowing in the two long sides. The field from a single long wire is $B = \frac{\mu_0 I}{2\pi s}$, so we need to integrate this from 0 to d and then multiply by 2 because the top and bottom wire both contribute to the flux in the same direction. The flux from one wire is thus $\Phi_1 = \frac{\mu_0 I}{2\pi}\int_0^d \frac{1}{s}l\,ds = \frac{\mu_0 Il}{2\pi}\ln s\Big|_0^d$. However, when we try to evaluate the natural log at 0 we find it diverges. So we introduce a small thickness ϵ to the wires, and integrate from ϵ to $d-\epsilon$. Thus $\Phi_1 = \frac{\mu_0 Il}{2\pi}\ln\frac{d-\epsilon}{\epsilon}$. We can ignore the ϵ in the numerator, because it is tiny compared to d, but it is very important in the denominator. Multiplying Φ_1 by 2 (for the two wires) and dividing by I gives us $L = \frac{\mu_0 l}{\pi}\ln(d/\epsilon)$. The size of the wire is very important in determining L!

3. **Griffiths 7.26**

   (a) The field inside a solenoid is $B = \mu_0 nI$, so the flux through a single turn of the wires is $\Phi_1 = \pi R^2 B = \pi R^2\mu_0 nI$. The total flux through a section of length l is the flux through one turn times the total number of turns, $N = nl$. Thus $\Phi = N\Phi_1 = \pi R^2\mu_0 n^2 lI \Rightarrow L = \pi R^2\mu_0 n^2 l$. Finally, $W = \frac{1}{2}LI^2 = \frac{1}{2}\pi R^2\mu_0 n^2 lI^2$.

   (b) $W = \frac{1}{2}\oint(\mathbf{A}\cdot\mathbf{I})\,dl$ where $\mathbf{A}(R) = (\mu_0 nI/2)R\,\hat{\boldsymbol{\phi}}$. We need to do the integral over the whole "loop", which we can do by integrating around a single turn and multiplying by the total number of turns, $N = nl$. So for a single turn, $W_1 = \frac{1}{4}\mu_0 nIR\oint\hat{\boldsymbol{\phi}}\cdot I\,\hat{\boldsymbol{\phi}}R\,d\phi = \frac{1}{4}\mu_0 nIR2\pi RI = \frac{1}{2}\pi\mu_0 nI^2R^2$. Multiplying by N gives $W = \frac{1}{2}\pi R^2\mu_0 n^2 lI^2$.

   (c) $W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2\,d\tau = \frac{1}{2\mu_0}\int_{\text{solenoid}}(\mu_0 nI)^2\,d\tau = \frac{1}{2\mu_0}\mu_0^2 n^2 I^2\pi R^2 l = \frac{1}{2}\pi R^2\mu_0 n^2 lI^2$.

   (d) $W = \frac{1}{2\mu_0}\left[\int_{\mathcal{V}} B^2 d\tau - \oint_{\mathcal{S}}(\mathbf{A}\times\mathbf{B})\cdot d\mathbf{a}\right]$. The volume is a cylindrical tube from radius $a < R$ to $b > R$. The only B-field is inside the cylinder, so the calculation of the first term is identical to part (c) but instead of the whole volume $\pi R^2 l$ we have only the volume for $s > a$, namely $\pi(R^2 - a^2)l$. So $\int B^2\,d\tau = \mu_0^2 n^2 I^2\pi(R^2 - a^2)l$. Now for the second term. Since $B = 0$ outside the solenoid, we only need to worry about the inner surface at $s = a$. $\mathbf{A}(a) = \frac{1}{2}\mu_0 nIa\,\hat{\boldsymbol{\phi}}$ and $\mathbf{B} = \mu_0 nI\,\hat{\mathbf{z}}$, so $\mathbf{A}\times\mathbf{B} = \frac{1}{2}\mu_0^2 n^2 I^2 a(\,\hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}}) = \frac{1}{2}\mu_0^2 n^2 I^2 a\,\hat{\mathbf{s}}$. Thus $\oint(\mathbf{A}\times\mathbf{B})\cdot d\mathbf{a} = \int(\frac{1}{2}\mu_0^2 n^2 I^2 a\,\hat{\mathbf{s}})\cdot[ad\phi\,dz(-\hat{\mathbf{s}})] = -\frac{1}{2}\mu_0^2 n^2 I^2 a^2 2\pi l$. Finally, $W = \frac{1}{2\mu_0}\left[\mu_0^2 n^2 I^2\pi(R^2 - a^2)l + \mu_0^2 n^2 I^2 a^2\pi l\right] = \frac{1}{2}\pi R^2\mu_0 n^2 lI^2$. All four methods agree!

4. **Griffiths 7.58**

   (a) The ribbon looks like a long, parallel plate capacitor. If there is surface charge $+\sigma$ on the top, and $-\sigma$ on the bottom, the field between the "plates" is $E = \sigma/\epsilon_0$, which means the potential difference between them is $V = Eh = \sigma h/\epsilon_0$. In a length L, the charge per area is $\sigma = Q/wL$. Thus $C = Q/V = \frac{Q}{Qh/\epsilon_0 wL} \Rightarrow \mathcal{C} = C/L = \epsilon_0 w/h$.

   (b) If there is uniform surface charge K flowing down the top ribbon and back up the bottom, they produce a B-field between the ribbons which is approximately uniform and points perpendicular to the current. Over a small amperian loop, $BL = \mu_0 KL \Rightarrow B = \mu_0 K = \mu_0 I/w$. The flux passing between the ribbons, in a length l (now measured along the length of the ribbons) is $\Phi = Bhl = \frac{\mu_0 I}{w} hl = LI \Rightarrow \mathcal{L} = L/l = \mu_0 h/w$. running width-wise

   (c) $\mathcal{LC} = \frac{\mu_0 h}{w}\frac{\epsilon_0 w}{h} = \mu_0\epsilon_0 = 1/c^2 = 1.11 \times 10^{-17}\ \mathrm{s^2/m^2}$. The speed of propagation is c.

   (d) When a dielectric is present, the capacitance is multiplied by the dielectric constant, ϵ_r: $\mathcal{C}' = \epsilon_r\mathcal{C} = \epsilon_r\epsilon_0 w/h = \epsilon w/h$. The inductance works the same way. With the magnetic material present, $H = K$, so $B = \mu H = \mu K$ instead of $\mu_0 K$. So we just replace ϵ_0 and μ_0 with ϵ and μ. $\mathcal{LC} = \mu\epsilon$. The propagation speed is $v = 1/\sqrt{\mu\epsilon}$.

5. **Griffiths 7.30**

   (a) Treating the small loops as magnetic dipoles, the magnetic field due to loop 1 is $\mathbf{B}_1 = \frac{\mu_0}{4\pi r^3}[3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{m}_1] = \frac{\mu_0 I_1}{4\pi r^3}[3((\mathbf{a}_1 \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{a}_1]$, where $\mathbf{r}$ is the vector from loop 1 to loop 2 and $\mathbf{m}_1 = I_1\mathbf{a}_1$. If loop two is very small, then the magnetic field is essentially constant over its area, and we have $\Phi_2 = \mathbf{B}_1 \cdot \mathbf{a}_2 = \frac{\mu_0 I_1}{4\pi r^3}[3(\mathbf{a}_1 \cdot \hat{\mathbf{r}})(\mathbf{a}_2 \cdot \hat{\mathbf{r}}) - \mathbf{a}_1 \cdot \mathbf{a}_2] = MI_1$. Thus $M = \frac{\mu_0}{4\pi r^3}[3(\mathbf{a}_1 \cdot \hat{\mathbf{r}})(\mathbf{a}_2 \cdot \hat{\mathbf{r}}) - \mathbf{a}_1 \cdot \mathbf{a}_2]$

   (b) We want to keep a constant current I_1 in loop 1. However, turning on a current in loop two causes an emf in loop 1: $\mathcal{E}_1 = -M\frac{dI_2}{dt}$. The induced emf does work at a rate $P = \frac{dW}{dt} = I_1\mathcal{E}_1$, so the work done per unit time against the induced emf is opposite this, $\left.\frac{dW}{dt}\right|_1 = -I_1\mathcal{E}_1 = MI_1\frac{dI_2}{dt}$. Since I_1 is assumed to be constant, we can integrate this equation to get the total work done. Since the current in loop 2 starts at zero and increases to a final value of I_2, we have $W_1 = MI_1I_2 = \frac{\mu_0}{4\pi r^3}[3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})(\mathbf{m}_2 \cdot \hat{\mathbf{r}}) - \mathbf{m}_1 \cdot \mathbf{m}_2]$. This is the total energy of interaction between the two loops, which is opposite in sign to the result in equation 6.35 in the text. The reason is that there we derived the interaction energy of two *fixed* dipoles. The only work we needed to do to assemble the system was move one dipole around in the field of the other dipole. But in the current problem, we also included the work necessary to maintain the dipole moment of one loop in the presence of the other. It is a funny coincidence that the only difference between the two calculations is a minus sign.

6. **Griffiths 7.31** The displacement current density is $\mathbf{J}_d = \epsilon_0\frac{\partial\mathbf{E}}{\partial t}$. If there is a surface charge density $+\sigma$ on the one side of the gap, and $-\sigma$ on the other side, the electric inside the gap is approximately like that of a parallel plate capacitor, namely uniform inside, zero outside, and with magnitude σ/ϵ_0. Then $J_d = \epsilon_0\frac{\partial}{\partial t}E = \frac{\partial\sigma}{\partial t} = I/A = I/(\pi a^2)$, pointing in the $\hat{\mathbf{z}}$ direction. Then if we draw an amperian loop with radius s in the gap, there is no conduction current flowing through the loop, and we can use the extended Ampere's law:

$$\oint \mathbf{B}\cdot d\mathbf{l} = B(s)2\pi s = \mu_0 I_{d_{\text{enc}}} = \mu_0\frac{I}{\pi a^2}\pi s^2 = \mu_0 I\frac{s^2}{a^2} \Rightarrow B = \frac{\mu_0 I s^2}{2\pi s a^2} \Rightarrow \mathbf{B} = \frac{\mu_0 I s}{2\pi a^2}\,\hat{\boldsymbol{\phi}}$$

7. **Griffiths 7.37** We place a parallel plate capacitor in the sea water and connect it to a voltage source. There will be some normal *conduction* current, $\mathbf{J}_c$, due to electrons in the sea water traveling from one plate to the other; but there will also be some displacement current, $\mathbf{J}_d$, due to the changing electric fields. First we find the conduction current. If the potential difference between the plates is V then the electric field is $E = V/d$

where d is the distance between the plates. Then $J_c = \sigma E = \frac{1}{\rho}E = \frac{V_0 \cos(2\pi\nu t)}{\rho d}$. On the other hand, treating the seawater as a linear dielectric, the displacement current is given by

$$J_d = \frac{\partial D}{\partial t} = \frac{\partial}{\partial t}(\epsilon E) = \epsilon \frac{\partial}{\partial t}\left[\frac{V_0 \cos(2\pi\nu t)}{d}\right] = \frac{\epsilon V_0}{d}[-2\pi\nu \sin(2\pi\nu t)].$$

We are only interested in determining which contribution is bigger, so we don't need to worry about the time dependence or the fact that the currents are out of phase; we just make the ration of their amplitudes:

$$\frac{J_c}{J_d} = \frac{V_0}{\rho d}\frac{d}{\epsilon V_0 2\pi\nu} = \frac{1}{2\pi\nu\epsilon\rho} = \left[2\pi(4\times 10^8)(81)(8.85\times 10^{12})(0.23)\right]^{-1} = 2.41$$

It is a good exercise to check that the units all cancel to leave a pure number.

8. **Griffiths 7.50** Since we are assuming that the voltmeters draw negilgible current, there is a single square circuit containing the two resistors surrounding the solenoid. The flux through the loop formed by the circuit is the flux inside the solenoid, $\Phi = \alpha t$. Thus the emf in the loop is $\mathcal{E} = -\frac{d\Phi}{dt} = -\alpha$, which drives a current $I = |\mathcal{E}|/R = \alpha/(R_1 + R_2)$ in the counter-clockwise direction. Meter 1 measures the voltage drop across R_1, namely $V_1 = IR_1 = \alpha R_1/(R_1 + R_2)$ (it is positive because V_b is the higher potential) and meter 2 reads $V_2 = -IR_2 = -\alpha R_2/(R_1 + R_2)$ (V_b is at a lower potential).

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Problem Set 9

1. Griffiths 10.3

2. Griffiths 10.5

3. Griffiths 10.7

4. Griffiths 8.1

5. Griffiths 9.11

6. Griffiths 9.13

7. At the rate of 1 card/sec, psychic Uri Geller (`http://skepdic.com/geller.html`) turns over each card in a deck. He communicates by "paranormal" means the identity of each card to his assistant, from whom he is shielded with respect to sound and visible light.

As a physicist, you consider all EM waves to be normal. To test the notion that Uri's talents defy the laws of physics, you resolve to design a shield that will prevent Uri from using any relevant EM frequency to communicate with his assistant.
(**a**) Roughly what minimum EM frequency must Uri use? (*Hint*: Consider that a 56 kbps modem operates over audio telephone frequencies.)
(**b**) Design a spherical shell, enclosing a volume of 1 m^3 for Uri's comfort, that will attenuate the EM waves generated by Uri's brain to $\approx 1/400 \approx e^{-6}$ of their original amplitude. Use the minimum EM frequency that you calculated in (a).
(**c**) How much does your shield weigh? (Try to design the lightest shield that will do the job. Does it help to use a ferromagnetic material?)

8. Show that the results in Griffiths Eq. (9.147) are equivalent to the familiar formulæ

$$\tilde{R} = \frac{Z_2 - Z_1}{Z_2 + Z_1}$$
$$\tilde{T} = \frac{2Z_2}{Z_2 + Z_1} \text{ , where}$$
$$Z \equiv \frac{\tilde{E}_0}{\tilde{H}_0} \text{ ,}$$
$$\tilde{R} \equiv \frac{\tilde{E}_{0_R}}{\tilde{E}_{0_I}} \text{ , and}$$
$$\tilde{T} \equiv \frac{\tilde{E}_{0_T}}{\tilde{E}_{0_I}} \text{ ,}$$

and where Z is the characteristic impedance of the medium, $\tilde{R}$ is the amplitude reflection coefficient, and $\tilde{T}$ is the amplitude transmission coefficient.

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Solution Set 9 (compiled by Daniel Larson)

1. **Griffiths 10.3** First we'll calculate the fields. $\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = 0 - \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\,\hat{\mathbf{r}}$. $\mathbf{B} = \nabla\times\mathbf{A} = -\frac{qt}{4\pi\epsilon_0}\nabla\times\frac{\hat{\mathbf{r}}}{r^2} = 0$. These fields should be familiar, because they are the fields of a stationary point charge q at the origin. Thus $\rho = q\delta^3(\mathbf{r})$ and $\mathbf{J} = 0$. If you took the divergence of $\mathbf{E}$ to find ρ you got zero, which is correct everywhere *except* at the origin, where $\mathbf{E}$ blows up and derivatives are ill defined. The delta-function at the origin is the function that has zero divergence everywhere except the origin, which is why we use it to represent the charge density for a point charge.

2. **Griffiths 10.5** We're given λ, so we can calculate the new potentials.
$$\mathbf{A}' = \mathbf{A} + \nabla\lambda = -\frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\,\hat{\mathbf{r}} - \frac{1}{4\pi\epsilon_0}qt\nabla\!\left(\frac{1}{r}\right) = -\frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\,\hat{\mathbf{r}} + \frac{1}{4\pi\epsilon_0}qt\left(\frac{1}{r^2}\,\hat{\mathbf{r}}\right) = 0.$$
$$V' = V - \frac{\partial\lambda}{\partial t} = 0 - \left(-\frac{1}{4\pi\epsilon_0}\frac{q}{r}\right) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}.$$
These are the more familiar potentials for the point charge of Problem 1.

3. **Griffiths 10.7** We want $\nabla\cdot\mathbf{A} = -\mu_0\epsilon_0\frac{\partial V}{\partial t}$, so first assume that it isn't true and then prove we can make a gauge transformation so that it becomes true. So assume that $\nabla\cdot\mathbf{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t} = \Phi$, where Φ is a function that we know. For any λ, the new vector potential is $\mathbf{A}' = \mathbf{A} + \nabla\lambda$, while $V' = V - \frac{\partial\lambda}{\partial t}$, and we want to require $\nabla\cdot\mathbf{A}' + \mu_0\epsilon_0\frac{\partial V'}{\partial t} = 0$. Note that both the vector potential $\mathbf{A}$ and the scalar potential V transform. Thus
$$\nabla\cdot\mathbf{A}' + \mu_0\epsilon_0\frac{\partial V'}{\partial t} = \nabla\cdot\mathbf{A} + \nabla^2\lambda + \mu_0\epsilon_0\frac{\partial V}{\partial -}\mu_0\epsilon_0\frac{\partial^2\lambda}{\partial t^2} = \Phi + \Box^2\lambda = 0 \;\Rightarrow\; \Box^2\lambda = -\Phi.$$
This equation is of the form (10.16 (i)), so assuming we can find a solution λ, we can make a gauge transformation using λ to get into Lorentz gauge.

If we choose $\lambda = \int_0^t V\,dt'$ then $V' = V - \frac{\partial\lambda}{\partial t} = V - V = 0$, so we can always find a gauge in which the scalar potential vanishes. This doesn't cause any problems, because the electric field gets contributions from both the scalar potential and vector potential, so for a given E-field we can choose $V = 0$ and still get the proper $\mathbf{E}$ from $\mathbf{A}$. However, if we have a non-zero B-field, then finding a gauge where $\mathbf{A} = 0$ would mean $\mathbf{B} = \nabla\times\mathbf{A} = 0$, which would be a contradiction. Thus we *cannot* in general find a gauge in which $\mathbf{A} = 0$.

4. **Griffiths 8.1**

   (a) From Example 7.13, we have a cable consisting of two concentric cylinders. The B-field between them is $\mathbf{B} = \frac{\mu_0 I}{2\pi s}\,\hat{\boldsymbol{\phi}}$. For linear charge density λ on the inner cylinder, the electric field between the cylinders is $\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 s}\,\hat{\mathbf{s}}$. Thus
$$\mathbf{s} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{\lambda I}{4\pi^2\epsilon_0 s^2}\,\hat{\mathbf{z}}.$$
   We want to find the power being transported down the cable, so we need to integrate $\mathbf{s}$ over a cross-section of the cable perpendicular to the z-axis. $\mathbf{s}$ is only nonzero between the cylinders.
$$P = \int \mathbf{s}\cdot d\mathbf{a} = \int_a^b S\,2\pi s\,ds = \frac{\lambda I}{2\pi\epsilon_0}\int_a^b \frac{ds}{s} = \frac{\lambda I}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right).$$
   To express this in terms of the potential difference V, look back at the solutions to Homework #2, Problem 2.39 where we found that $V = |V(b) - V(a)| = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right)$. Substituting into our result, this gives $P = IV$.

(b) We proceed exactly as in part (a). For surface charge σ on the ribbons, $\mathbf{E} = \frac{\sigma}{\epsilon_0}\,\hat{\mathbf{z}}$. Also, $\mathbf{B} = \mu_0 K\,\hat{\mathbf{x}} = \frac{\mu_0 I}{w}\,\hat{\mathbf{x}}$ (see the solution to problem 7.58). Thus $\mathbf{s} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{\sigma I}{\epsilon_0 w}\,\hat{\mathbf{y}}$. Again, the power transported is the surface integral of $\mathbf{s}$ over a cross-section perpendicular to the length of the ribbons. In this case, $\mathbf{s}$ is constant so we just multiply by the area, wh. Thus $P = \sigma I h/\epsilon_0$. But the potential difference is $V = -\int \mathbf{E}\cdot d\mathbf{l} = \sigma h/\epsilon_0$, which again gives $P = IV$.

5. **Griffiths 9.11** We want to compute the time average of $f = A\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \delta_a)$ multiplied by $g = B\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \delta_b)$ over one period, T. First, lets do it the long way.

$$\langle fg\rangle = \frac{1}{T}\int_0^T A\cos(\mathbf{k}\cdot\mathbf{r}-\omega t+\delta_a)\,B\cos(\mathbf{k}\cdot\mathbf{r}-\omega t+\delta_b)dt = \frac{AB}{2T}\int_0^T [\cos(2\mathbf{k}\cdot\mathbf{r} - 2\omega t + \delta_a + \delta_b) + \cos(\delta_a - \delta_b)]\,dt.$$

Here I used the trig identity: $\cos(\alpha)\cos(\beta) = \frac{1}{2}(\cos(\alpha+\beta) + \cos(\alpha-\beta))$. Now the integral isn't too hard. The first term is the integral of a cosine over one full period, which gives zero. The second term is independent of t, so just gets multiplied by T. Thus $\langle fg\rangle = \frac{AB}{2T}\cos(\delta_a - \delta_b)T = \frac{1}{2}AB\cos(\delta_a - \delta_b)$.

Now let's calculate the average using complex notation. We let $f = \mathrm{Re}(\tilde{f})$ where $\tilde{f} = Ae^{i(\mathbf{k}\cdot r - \omega t + \delta_a)} = Ae^{i\delta_a}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \equiv \tilde{A}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$. Similarly, $g = \mathrm{Re}(\tilde{g})$ where $\tilde{g} = \tilde{B}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ and $\tilde{B} = Be^{i\delta_b}$. Then

$$\begin{aligned}\frac{1}{2}\mathrm{Re}(\tilde{f}\tilde{g}^*) &= \frac{1}{2}\mathrm{Re}\Big(\tilde{A}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\tilde{B}^* e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\Big) = \frac{1}{2}\mathrm{Re}\big(Ae^{i\delta_a}Be^{-i\delta_b}\big)\\ &= \frac{AB}{2}\mathrm{Re}\Big(ABe^{i(\delta_a-\delta_b)}\Big) = \frac{AB}{2}AB\cos(\delta_a - \delta_b) = \langle fg\rangle.\end{aligned}$$

6. **Griffiths 9.13** The derivation of the exact reflection and transmission coefficients, as defined by Griffiths, follows Section 9.3.2 in the text up through equation (9.82).

$$\tilde{E}_{0_R} = \left(\frac{1-\beta}{1+\beta}\right)\tilde{E}_{0_I}, \qquad \tilde{E}_{0_T} = \left(\frac{2}{1+\beta}\right)\tilde{E}_{0_I}$$

where $\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}$. Griffiths defines the intensity reflection coefficient as the ratio of reflected to incident intensity.

$$R \equiv \frac{I_R}{I_I} = \left(\frac{E_{0_R}}{E_{0_I}}\right)^2 = \left(\frac{1-\beta}{1+\beta}\right)^2 = \left(\frac{\mu_2 v_2 - \mu_1 v_1}{\mu_2 v_2 - \mu_1 v_1}\right)^2 = \left(\frac{\mu_2 n_1 - \mu_1 n_2}{\mu_2 n_1 - \mu_1 n_2}\right)^2$$

Notice that $\mu_i\epsilon_i c^2 = n_i^2$ (Eqn 9.68). This implies $\frac{\epsilon_2 v_2}{\epsilon_1 v_1} = \frac{n_2^2\mu_1 v_2}{\mu_2 n_1^2 v_1} = \frac{n_2^2\mu_1 n_1}{n_1^2\mu_2 n_2} = \frac{\mu_1 n_2}{\mu_2 n_1} = \beta$. Thus the intensity transmission coefficient is

$$T \equiv \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left(\frac{E_{0_T}}{E_{0_I}}\right)^2 = \beta\left(\frac{2}{1+\beta}\right)^2$$

You can plug in β in terms of μ and n, but that isn't too enlightening. To add the two coefficients it is easiest to leave things in terms of β.

$$T + R = \frac{4\beta}{(1+\beta)^2} + \frac{(1-\beta)^2}{(1+\beta)^2} = \frac{1}{(1+\beta)^2}\big(4\beta + (1-\beta)^2\big) = \frac{1}{(1+\beta)^2}\big(4\beta + 1 - 2\beta + \beta^2\big) = \frac{(1+2\beta+\beta^2)}{(1+\beta)^2} = 1$$

7. **Psychic**

(a) If Uri is flipping 1 card per second then he must transfer information at a rate of about 6 bits per second, since 6 bits provides $2^6 = 64$ different combinations, which is enough to specify a single card out of the 52 cards in a deck. Knowing that a 56 kbps modem essentially saturates the capacity of the phone network, which carries frequencies between 200 Hz and 3000 Hz, we can assume that to carry 6 bps one would need a bandwidth of:

$$\frac{6\ \text{bps}}{56000\ \text{bps}}(3000 - 200)\ \text{Hz} = 0.3\ \text{Hz}$$

To get this bandwidth, the minimum frequency Uri would need will be about 0.3 Hz $\Rightarrow$ $\omega = 1.88\ \text{s}^{-1}$.

(b) If we make a spherical shell enclosing a cubic meter, it must have an inner radius of $R = (3/4\pi)^{1/3} = 0.62$ m. We want to make it out of a conducting material so that the EM waves are attenuated as they travel through the shell. From equation (9.127) in the text, we see that the amplitude is proportional to $e^{-\kappa z}$, where z is the direction of propagation. So to get the amplitude attenuated by $1/400 \approx e^{-6}$ we need a thickness $t = 6/\kappa$. From equation (9.126),

$$\kappa \equiv \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right]^{1/2}$$

To make a small shell, we need κ to be large, which means choosing a material with a high conductivity. For example, let's use copper. (It has a high conductivity, but lower density and lower cost than silver.) From the table section 7.1, the conductivity of copper is $\sigma = 1/\rho = 6.0\times 10^7 \quad (\Omega\text{m})^{-1}$. Assuming $\epsilon \approx \epsilon_0$ and $\mu \approx \mu_0$, we find $\kappa = 8.4$ m^{-1}, so the thickness we would need would be $t = 6/\kappa = 0.71$ m. That's quite a thick shell!

(c) Since the inner radius of the shell is 0.62 m, the outer radius would have to be 1.33 m, which gives a total volume of copper $V = \frac{4}{3}\pi\left(1.33^3 - 0.62^3\right) = 8.86$ m^3. The density of copper is 8.96 g/cm^3 = 8.96 $\times 10^3$ kg/m^3, which gives a total mass of $(8.86)(8.96\times 10^3) = 79000$ kg! That's about 87 tons. So copper probably isn't the best material to use.

The suggestion to use a ferromagnetic material is a good one. Looking at the table of resistivities, we see that iron has a conductivity $\sigma \approx 10^7$, only a factor of 6 smaller than copper. However, iron can also have permeability μ much bigger than μ_0, which will help increas κ and thus decrease the required shell thickness. A further bonus is that the density of iron is slightly smaller than that of copper. In my brief research on the subject (namely typing "magnetic permeability" into Google), I found the claim that well designed ferromagnets can have permeabilities up to $\mu = 10^6\mu_0$. Taking this extreme case, and still assuming that $\epsilon \approx \epsilon_0$, we find $\kappa = 3510$ m^{-1}. Thus the thickness is $t = 6/\kappa = 1.7$ mm. Now the shell requires only $V = 4\pi R^2 t = 8.2\times 10^3$ cm^3 of iron. With a density of 7.86 g/cm^3, this yields a shield that weighs about 64 kg. Still, this isn't light (about 142 lbs), but it is much more reasonable than the gargantuan copper shield. Of course, we're not likely to find that much iron with so high a permeability, but this demonstrates how important ferromagnetic materials are for practical shielding.

8. **Compare Results** Griffiths has equation (9.147):

$$\tilde{E}_{0_R} = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\tilde{E}_{0_I}, \quad \tilde{E}_{0_T} = \left(\frac{2}{1+\tilde{\beta}}\right)\tilde{E}_{0_I} \quad \Rightarrow \quad \tilde{R} \equiv \frac{\tilde{E}_{0_R}}{\tilde{E}_{0_I}} = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right), \text{ and } \tilde{T} \equiv \frac{\tilde{E}_{0_T}}{\tilde{E}_{0_I}} = \left(\frac{2}{1+\tilde{\beta}}\right)$$

Now we need to relate $\tilde{\beta}$ to Z_1 and Z_2. Griffiths defines $\tilde{\beta} = \frac{\mu_1 v_1}{\mu_2\omega}\tilde{k}_2$, while we define $Z = \frac{\tilde{E}_0}{\tilde{H}_0}$ in each medium. Since $B = \mu H$ in linear materials, and using Griffiths's equations (9.140)-(9.142), we find:

$$Z_1 = \frac{\tilde{E}_{0_I}}{\tilde{H}_{0_I}} = \frac{\tilde{E}_{0_I}}{\frac{1}{\mu_1}\tilde{B}_{0_I}} = \mu_1 v_1 \quad \text{and} \quad Z_2 = \frac{\tilde{E}_{0_T}}{\tilde{H}_{0_T}} = \frac{\tilde{E}_{0_T}}{\frac{1}{\mu_2}\tilde{B}_{0_T}} = \frac{\mu_2\omega}{\tilde{k}_2}$$

Thus $\tilde{\beta} = \frac{\mu_1 v_1}{\mu_2\omega}\tilde{k}_2 = Z_1/Z_2$. Plugging this into the above formulas and multiplying both numerator and denominator by Z_2 we get the "familiar" equations:

$$\tilde{R} = \frac{Z_2 - Z_1}{Z_2 + Z_1} \quad \text{and} \quad \tilde{T} = \frac{2Z_2}{Z_2 + Z_1}$$

University of California, Berkeley
Physics 110A Fall 2001 Section 1 (*Strovink*)

Problem Set 10

1. Griffiths 9.19

2. Griffiths 9.20

3. Griffiths 9.21

4. Griffiths 11.3

5. Griffiths 11.4. You need calculate only the **time average** Poynting vector, intensity, and total power radiated (this is much simpler than computing the full time-dependent expressions).

6. Griffiths 11.9

7. Griffiths 11.14

8. Griffiths 11.21

University of California, Berkeley
Physics 110A Fall 2001 Section 1 (*Strovink*)

Solution Set 10 (compiled by Daniel Larson)

1. **Griffiths 9.19**

(a) For a poor conductor, $\sigma \ll \omega\epsilon$, we can expand the second square root in the formula for κ (Eqn 9.126):

$$\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2} \approx 1 + \frac{1}{2}\left(\frac{\sigma}{\epsilon\omega}\right)^2 + \cdots.$$

Then we can evaluate κ explicitly:

$$\kappa = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[1+\frac{1}{2}\left(\frac{\sigma}{\epsilon\omega}\right)^2 - 1\right]^{1/2} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} \Rightarrow d = \frac{1}{\kappa} = \frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}$$

For pure water, $\epsilon = 80.1\epsilon_0$ (Table 4.2), $\mu = \mu_0(1+\chi_m) \approx \mu_0$ (Table 6.1), and $\sigma = 1/\rho = 1/(2.5 \times 10^5)$ (Table 7.1). Plugging these numbers in, you should find $d = 1.19 \times 10^4$ m.

(b) For a good conductor, $\sigma \gg \omega\epsilon$, so we can ignore the "1s" in Eqn. 9.126. In particular, notice that in this limit $\kappa = k$, which means $d = 1/\kappa = 1/k = 1/(2\pi/\lambda) = \lambda/2\pi$. In this limit we have:

$$\kappa \approx \omega\sqrt{\frac{\epsilon\mu}{2}}\left(\frac{\sigma}{\epsilon\omega}\right)^{1/2} = \sqrt{\frac{\omega\mu\sigma}{2}} = 8 \times 10^7 \text{ m}^{-1},$$

where I've used the given values. So the skin-depth is $d = 1/\kappa = 1.3 \times 10^-8$ m $= 13$ nm.

(c) We're still in the regime where $\kappa \approx k$, so the phase difference is $\phi = \tan^{-1}(\kappa/k) = tan^{-1}1 = 45°$, and it is always the magnetic field that lags behind. The ratio of their amplitudes is given by Eqn. 9.137:

$$\frac{B_0}{E_0} = \sqrt{\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\mu}\right)^2}} \approx \sqrt{\frac{\mu\sigma}{\omega}} = 10^{-7} \text{ s/m}.$$

(I used the same numbers as given for part (b).) Compared to the ratio of the amplitudes in vacuum, namely $1/c$, the B-field is comparatively about 100 times larger in the good conductor.

2. **Griffiths 9.20**

(a) Using Eqn (9.138), and taking the time average so $\cos^2 \to \frac{1}{2}$:

$$\begin{aligned} u &= \frac{1}{2}\left(\epsilon E^2 + \frac{1}{\mu}B^2\right) = \frac{1}{2}e^{-2\kappa z}\left[\epsilon E_0^2\frac{1}{2} + \frac{1}{\mu}B_0^2\frac{1}{2}\right] = \frac{1}{4}e^{-2\kappa z}E_0^2\left[\epsilon + \frac{1}{\mu}\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}\right] \\ &= \frac{\epsilon}{4}E_0^2 e^{-2\kappa z}\left[1+\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}\right] = \frac{\epsilon}{4}E_0^2 e^{-2\kappa z}\frac{2}{\epsilon\mu}\frac{k^2}{\omega^2} = \frac{k^2}{2\mu\omega^2}E_0^2 e^{-2\kappa z} \end{aligned}$$

(Here I used (9.126) to replace the square root with k.) We can see that the magnetic contribution dominates by looking at the first expression in the second line, above. The 1 represents the electric contribution, while the $\sqrt{1+(\sigma/\epsilon\omega)^2}$ comes from the magnetic contribution. Since the second term is always greater than or equal to 1, the magnetic contribution dominates.

(b) You can do this problem by calculating $\mathbf{S}$ and time averaging, but I prefer to do it in a more physical way. From (9.138) we can see that the energy is flowing in the $\hat{\mathbf{z}}$ direction, since $\mathbf{S} \sim \mathbf{E} \times \mathbf{B} \sim \hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$. So let's consider a region of area A in the xy-plane. The energy density below the plane was calculated in (a), and is moving upward ($+z$) at a speed $v = \omega/k$. The amount of energy that will pass through the area in a time Δt is then $U = uAv\Delta t$. This is the amount of energy contained in a box with cross sectional area A and length $v\Delta t$. The intensity is the energy per unit area per unit time, so $I = U/(A\Delta t) = uv = u\omega/k = \frac{k}{2\mu\omega}E_0^2 e^{-2\kappa z}$.

3. **Griffiths 9.21** Using (9.147) we find

$$R = \left|\frac{\tilde{E}_{0_R}}{\tilde{E}_{0_I}}\right|^2 = \left|\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right|^2 = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\left(\frac{1-\tilde{\beta}^*}{1+\tilde{\beta}^*}\right) \quad \text{with} \quad \tilde{\beta} = \frac{\mu_1 v_1}{\mu_2 \omega}\tilde{k}_2$$

Since silver is a good conductor, $\sigma \gg \epsilon\omega$, so as in Problem 9.19(b) above, we have $k \approx \kappa$ in the silver, so $\tilde{k}_2 = k_2 + i\kappa_2 \approx \kappa_2(1+i) = \sqrt{\sigma\omega\mu_2/2}(1+i)$. Then $\tilde{\beta} = \mu_1 v_1 \sqrt{\sigma/2\mu_2\omega}(1+i) \equiv \alpha(1+i)$, where α is collection of constant, but in particular, is a real number. Then we have

$$R = \left(\frac{1-\alpha-i\alpha}{1+\alpha+i\alpha}\right)\left(\frac{1-\alpha+i\alpha}{1+\alpha-i\alpha}\right) = \frac{(1-\alpha)^2+\alpha^2}{(1+\alpha)^2+\alpha^2}$$

Now we need to evaluate α with the given numbers. $\alpha = \mu_1 v_1\sqrt{\sigma/2\mu_2\omega} = \mu_0 c\sqrt{\sigma/2\mu_0\omega} = 29$. Plugging this into the previous expression we find $R = 0.93$. So 93% of the light is reflected.

4. **Griffiths 11.3** In the text we calculated the total power radiated by a dipole: $\langle P\rangle = \dfrac{\mu_0 p_0^2\omega^4}{12\pi c}$. If instead there were a wire connecting the ends of the dipole, the power dissipated as heat would be $P = I^2R$. The current we need to use is the current produced by the oscillating dipole, namely $I = -q_0\sin(\omega t)$. Thus $P = q_0^2\omega^2\sin^2(\omega t)R$. Taking the time average, the sine just gives a factor of $\frac{1}{2}$, so $\langle P\rangle = \frac{1}{2}q_0^2\omega^2 R$. Setting this equal to the previous expression for $\langle P\rangle$, recalling that $p_0 = q_0 d$, we find $R = \dfrac{\mu_0\omega^2 d^2}{6\pi c} = \dfrac{2}{3}\pi\mu_0 c\left(\dfrac{d}{\lambda}\right)^2$, where I used $\omega = 2\pi c/\lambda$. Evaluating this with $\mu_0 = 4\pi\times 10^{-7}$ N/A^2 and $c = 3\times 10^8$ m/s, we get $R = 80\pi^2(d/\lambda)^2\ \Omega = 790(d/\lambda)^2\ \Omega$. For the station 101.1 FM, the frequency is 101.1 MHz, so since $c = \lambda f$, we find $\lambda \approx 3$ m. Thus $d/\lambda = .005/3$, so $R = 0.22\ \Omega$. For AM waves the wavelength is longer, so the radiation resistance is even smaller.

5. **Griffiths 11.4** We want to find the radiation due to a single rotating electric dipole, which can be represented by the superposition of two other electric dipoles, $\mathbf{p}_1 = p_0\cos(\omega t)\,\hat{\mathbf{x}}$ and $\mathbf{p}_2 = p_0\sin(\omega t)\,\hat{\mathbf{y}}$. In the text we derived the fields and power due to a single electric dipole oscillating along the z-axis. The superposition principle guarantees that we can simply add the electric and magnetic fields from two different sources, but it doesn't tell us whether we can add the power from the two sources (in general we *cannot* add powers). So let's determine the fields, since we can use superposition on them.

   Equation (11.18) in the text gives the E-field for a single electric dipole oscillating along the z-axis. To most easily use this formula to handle dipoles oscillating along $\hat{\mathbf{x}}$ or $\hat{\mathbf{y}}$, let's write $\sin\theta\,\hat{\boldsymbol{\theta}}$ in a different way. Since θ is measured from the z-axis, you can show $\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$. (Remember that $\hat{\boldsymbol{\theta}}$ and $\hat{\mathbf{r}}$ change depending on where you are in space. So pick a point and draw all three vectors $\hat{\mathbf{z}}$, $\hat{\mathbf{r}}$, and $\hat{\boldsymbol{\phi}}$ with their tails on that point.) Since $\cos\theta = z/r$, this means: $\sin\theta\,\hat{\boldsymbol{\theta}} = \frac{z}{r}\,\hat{\mathbf{r}} - \hat{\mathbf{z}}$. So now the formula representing a dipole oscillating along the x-axis will simply have x replacing z. For $\mathbf{p}_2$, we have y replacing z and also the time dependence also switches from cos to sin. Thus making the proper modifications to (11.18) we have:

$$\begin{aligned}\mathbf{E}_{\text{tot}} &= \mathbf{E}_1 + \mathbf{E}_2 = \frac{\mu_0 p_0\omega^2}{4\pi r}\cos(\omega t_0)\left(\frac{x}{r}\,\hat{\mathbf{r}} - \hat{\mathbf{x}}\right) + \frac{\mu_0 p_0\omega^2}{4\pi r}\sin(\omega t_0)\left(\frac{y}{r}\,\hat{\mathbf{r}} - \hat{\mathbf{y}}\right)\\ &= \frac{\mu_0 p_0\omega^2}{4\pi r}\left[\cos(\omega t_0)\left(\frac{x}{r}\,\hat{\mathbf{r}} - \hat{\mathbf{x}}\right) + \sin(\omega t_0)\left(\frac{y}{r}\,\hat{\mathbf{r}} - \hat{\mathbf{y}}\right)\right]\end{aligned}$$

   where I've used the shorthand $t_0 = t - r/c$. Now, we could do the same thing for $\mathbf{B}$, but it is simpler to notice that $\mathbf{B} = \frac{1}{c}(\,\hat{\mathbf{r}}\times\mathbf{E}_{\text{tot}})$. We can then more easily calculate the Poynting vector.

$$S = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{1}{\mu_0 c}(\mathbf{E}\times(\,\hat{\mathbf{r}}\times\mathbf{E})) = \frac{1}{\mu_0 c}\left(E^2\,\hat{\mathbf{r}} - (\mathbf{E}\cdot\hat{\mathbf{r}})\mathbf{E}\right) = \frac{1}{\mu_0 c}E_{\text{tot}}^2\,\hat{\mathbf{r}}$$

   Here we used that fact that $\mathbf{E}\cdot\hat{\mathbf{r}} = 0$ because $\left(\frac{x}{r}\,\hat{\mathbf{r}} - \hat{\mathbf{x}}\right)\cdot\hat{\mathbf{r}} = \frac{x}{r} - \frac{x}{r} = 0$, and the same holds for the other piece of $\mathbf{E}_{\text{tot}}$. So we just need to calculate $E_{\text{tot}}^2 = (\mathbf{E}_1+\mathbf{E}_2)\cdot(\mathbf{E}_1+\mathbf{E}_2) = E_1^2 + E_2^2 + 2\mathbf{E}_1\cdot\mathbf{E}_2$. Here's where the time averaging helps. All the time dependence is in the trig functions. The three terms in E_{tot}^2 are proportional to

$\cos^2(\omega t_0)$, $\sin^2(\omega t_0)$, and $\cos(\omega t_0) sin(\omega t_0)$ respectively. The $\cos^2$ and $\sin^2$ both average to $\frac{1}{2}$, while the cross term averages to 0. So now we have

$$\langle S\rangle = \frac{1}{\mu_0 c}\left(\frac{\mu_0 p_0 \omega^2}{4\pi r}\right)^2 \left[\frac{1}{2}\left(\frac{x}{r}\,\hat{\mathbf{r}} - \hat{\mathbf{x}}\right)^2 + \frac{1}{2}\left(\frac{y}{r}\,\hat{\mathbf{r}} - \hat{\mathbf{y}}\right)^2\right]\hat{\mathbf{r}}$$

Working out the term in square brackets we get

$$\frac{1}{2}\left(\frac{x}{r}\,\hat{\mathbf{r}} - \hat{\mathbf{x}}\right)^2 + \frac{1}{2}\left(\frac{y}{r}\,\hat{\mathbf{r}} - \hat{\mathbf{y}}\right)^2 = \frac{1}{2}\left(\frac{x^2}{r^2} - 2\frac{x^2}{r^2} + 1 + \frac{y^2}{r^2} - 2\frac{y^2}{r^2} + 1\right) = 1 - \frac{1}{2}\frac{x^2+y^2}{r^2} = 1 - \frac{1}{2}\sin^2\theta$$

Putting this back, we get the final result

$$\langle S\rangle = \frac{\mu_0}{c}\left(\frac{p_0\omega^2}{4\pi r}\right)^2\left(1 - \frac{1}{2}\sin^2\theta\right)\hat{\mathbf{r}}$$

The intensity profile is an ellipsoid centered at the origin which is twice as long in the z-direction as in the x and y directions.

To find the total power we integrate the intensity over a spherical surface.

$$\begin{aligned} P &= \int \langle S\rangle \cdot d\mathbf{a} = \frac{\mu_0}{c}\left(\frac{p_0\omega^2}{4\pi}\right)^2 \int \frac{1}{r}\left(1 - \frac{1}{2}\sin^2\theta\right) r^2 \sin\theta\, d\theta d\phi = \frac{\mu_0 p_0^2\omega^4}{16\pi^2 c} 2\pi\left[\int_0^\pi \sin\theta\, d\theta - \frac{1}{2}\int_0^\pi \sin^3\theta\, d\theta\right] \\ &= \frac{\mu_0 p_0^2 \omega^4}{8\pi c}\left(2 - \frac{1}{2}\frac{4}{3}\right) = \frac{\mu_0 p_0^2\omega^4}{6\pi c} \end{aligned}$$

Note that this is in fact just twice the power emitted by a single oscillating dipole. We could have just added the power from both dipoles in this case because they were out of phase, which made the cross term (i.e. $\sin(\omega t)\cos(\omega t)$) average to zero.

6. **Griffiths 11.9** The ring possesses an electric dipole moment which is rotating. There might also be a changing magnetic dipole moment, but that contribution is much smaller than the contribution from the electric dipole radiation. (See the discussion at the end of Section 11.1.3.) So we need to first calculate the ring's electric dipole moment. Let's calculate it at a fixed time, $t = 0$, so it isn't moving while we integrate.

$$\begin{aligned} \mathbf{p} &= \int \rho(\mathbf{r}')\mathbf{r}'\, d\tau = \int \lambda(\mathbf{r})\mathbf{r}\, dl = \int (\lambda_0 \sin\phi)(b\sin\phi\,\hat{\mathbf{y}} + b\cos\phi\,\hat{\mathbf{x}})\, b\, d\phi \\ &= \lambda_0 b^2\left(\hat{\mathbf{y}}\int_0^{2\pi} \sin^2\phi\, d\phi + \hat{\mathbf{x}}\int_0^{2\pi} \sin\phi\, \cos\phi\, d\phi\right) = \lambda_0 b^2(\pi\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{x}}) = \pi b^2\lambda_0\,\hat{\mathbf{y}} \end{aligned}$$

Now we can use the result from the previous problem to find the power radiated by this rotating dipole. We have $p_0 = \pi b^2 \lambda_0$. Thus $P = \mu_0 \pi b^4 \lambda_0^2 \omega^4/(6c)$.

7. **Griffiths 11.14** Treating the hydrogen atom classically, we have an electron orbiting around the proton due to their electrostatic attraction. Thus we can find the velocity of the electron by setting the Coulomb force equal to the centripetal force needed to keep the electron in a circular orbit.

$$\mathbf{F}_{\text{Coul}} = \mathbf{F}_{\text{cent}} \;\Rightarrow\; \frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2} = m_e\frac{v^2}{r} \;\Rightarrow\; v = \sqrt{\frac{e^2}{4\pi\epsilon_0 m_e r}}$$

For an initial radius of $r = a_0 = 5\times 10^{-11}$ m, you can plug in numbers and should find $v/c = 0.0075$. Since v depends on the square-root of r, when r is 100 times smaller, v/c will only be 10 times bigger, which is still pretty small. So it is indeed a good approximation to assume that the electron is non-relativistic for most of its trip.

Because of the centripetal acceleration, $a = v^2/r$, the electron will radiate. Using the Larmor formula,

$$P = \frac{\mu_0 e^2}{6\pi c}\left(\frac{v^2}{r}\right)^2 = \frac{\mu_0 e^2}{6\pi c}\left(\frac{1}{4\pi\epsilon_0}\frac{e^2}{mr^2}\right)^2 = \frac{e^2}{6\pi\epsilon_0 c^3}\left(\frac{1}{4\pi\epsilon_0}\frac{e^2}{mr^2}\right)^2.$$

As the electron spirals in, it gains kinetic energy and loses potential energy, and also loses some energy to radiation. For the total (kinetic plus potential plus radiated) to be conserved, the decrease in the kinetic plus potential energy must equal the power radiated.

$$U = U_{KE} + U_{PE} = \frac{1}{2}mv^2 - \frac{1}{4\pi\epsilon_0}\frac{e^2}{r} = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{e^2}{r} - \frac{1}{4\pi\epsilon_0}\frac{e^2}{r} = -\frac{1}{8\pi\epsilon_0}\frac{e^2}{r}$$

Then

$$-\frac{dU}{dt} = -\frac{1}{8\pi\epsilon_0}\frac{e^2}{r^2}\frac{dr}{dt} = P = \frac{e^2}{6\pi\epsilon_0 c^3}\left(\frac{1}{4\pi\epsilon_0}\frac{e^2}{mr^2}\right)^2 \Rightarrow \frac{dr}{dt} = -\frac{4}{3c^3}\left(\frac{1}{4\pi\epsilon_0}\frac{e^2}{m}\right)^2\frac{1}{r^2} \equiv -\frac{A}{r^2}$$

Now we can integrate to find the total time.

$$-r^2\,dr = A\,dt \Rightarrow -\int_{a_0}^{0} r^2\,dr = \int_0^T A\,dt \Rightarrow \left.\frac{r^3}{3}\right|_0^{a_0} = \frac{a_0^3}{3} = AT \Rightarrow T = \frac{a_0^3}{3}\frac{3c^3}{4}\left(\frac{4\pi\epsilon_0 m}{e^2}\right)^2 = a_0^3 c\left(\frac{2\pi\epsilon_0 mc}{e^2}\right)^2$$

Plugging in the values of the constants and double checking that the units work out, you should find $T = 1.3 \times 10^{-11}$ s. Clearly this theory has a problem, because we know the hydrogen atom lasts much longer than that! This is one reason quantum mechanics was needed.

8. **Griffiths 11.21**

(a) We have an oscillating electric dipole with magnitude $p_0 = qd$. The frequency of oscillation is $\omega = \sqrt{k/m}$. The time averaged Poynting vector is $\langle \mathbf{S} \rangle = \left(\frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\right)\frac{\sin^2\theta}{r^2}\,\hat{\mathbf{r}}$. Let's choose our z-axis to be pointing down, towards the floor. The radiation is traveling radially away from our radiating dipole, which means it hits the floor with some angle at radius R. We can get the power per unit area on the floor by dotting $\mathbf{S}$ with $\hat{\mathbf{z}}$, the normal to the floor, which gives us a factor of $\cos\theta$, where θ is the angle between the vertical line from the dipole to the floor and the radial line from the dipole to the point at radius R. From this right triangle we find $\sin\theta = R/r$, $\cos\theta = h/r$, and $r^2 = R^2 + h^2$. So we have:

$$I_{\text{floor}} = \langle \mathbf{S} \rangle \cdot \hat{\mathbf{z}} = \left(\frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\right)\frac{\sin^2\theta\cos\theta}{r^2} = \left(\frac{\mu_0 q^2 d^2 \omega^4}{32\pi^2 c}\right)\frac{R^2 h}{r^5} = \left(\frac{\mu_0 q^2 d^2 \omega^4}{32\pi^2 c}\right)\frac{R^2 h}{(R^2+h^2)^{5/2}}$$

To find the location of most intense radiation, we take a derivative of I with respect to R and set it equal to zero.

$$\frac{dI}{dR} = 0 \quad \Rightarrow \quad \frac{d}{dR}\left[\frac{R^2}{(R^2+h^2)^{5/2}}\right] = 0 \Rightarrow \frac{2R}{(R^2+h^2)^{5/2}} - \frac{5}{2}\frac{2RR^2}{(R^2+h^2)^{7/2}} = 0 \Rightarrow R = 0$$

$$\text{or} \quad (R^2+h^2) = \frac{5}{2}R^2 \Rightarrow R^2 = \frac{2}{3}h^2$$

Since the intensity is zero for $R = 0$, and falls off to zero as $R \to \infty$, there must indeed be a maximum for $R = \sqrt{2/3}h$.

(b) Now we want to find the average energy per unit time (i.e. power) striking an infinite floor. So we need to integrate I over all R from 0 to ∞.

$$P = \int I(R)\,da = \int_0^\infty I(R) 2\pi R\,dR = 2\pi\left(\frac{\mu_0 q^2 d^2 \omega^4}{32\pi^2 c}\right) h \int_0^\infty \frac{R^3\,dR}{(R^2+h^2)^{5/2}}$$

This integral can be done using a trig substitution, $R^2 = h^2 \tan^2 u$.

$$\int_0^\infty \frac{R^3\,dR}{(R^2+h^2)^{5/2}} = \frac{1}{h}\int_0^{\pi/2}\frac{\tan^3 u}{\sec^3 u}\,du = \frac{1}{h}\int_0^{\pi/2}\sin^3 u\,du = \frac{2}{3h}$$

Thus the total power that hits the floor is $P = \frac{\mu_0 q^2 d^2 \omega^4}{24\pi c}$. As you would expect, this is half of the total power radiated by an electric dipole (compare Eqn. 11.22), because the other half of the radiation hits the ceiling.

(c) If the amplitude of oscillation is $x_0(t)$, then the total energy of the simple harmonic oscillator at any time is $U = \frac{1}{2}kx_0(t)^2$. (So $x_0(0) = d$.) The decrease in energy should be equal to the total power radiated, which is twice the result found in part b.

$$\begin{aligned} -\frac{dU}{dt} &= -\frac{1}{2}k\frac{d}{dt}(x_0(t)^2) = 2P = \frac{\mu_0 q^2 \omega^4}{12\pi c}x_0(t)^2 \;\Rightarrow\; \frac{d}{dt}(x_0(t)^2) = -\frac{\mu_0 q^2 \omega^4}{6\pi kc}(x_0(t)^2) = -A(x_0(t)^2) \\ &\Rightarrow\; x_0(t)^2 = d^2 e^{-At} \;\Rightarrow\; x_0(t) = d^2 e^{-At/2} \end{aligned}$$

Thus when $t = 2/A$, we'll have $x_0 = de^{-1} = d/e$. Thus $\tau = \frac{12\pi kc}{\mu_0 q^2 \omega^4} = \frac{12\pi cm^2}{\mu_0 q^2 k}$. (Remember, $\omega = \sqrt{k/m}$.)

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Problem Set 11

1. The electric vector of a fully polarized plane EM wave is given by the expression

$$\mathbf{E} = E_0\big[\hat{x}\cos(kz-\omega t) + \hat{y}\, b\cos(kz-\omega t+\phi)\big]\ ,$$

where E_0, k, ω, b, and ϕ are real constants.
(**a**)
Defining

$$\mathbf{E} \equiv \mathrm{Re}\big(\tilde{\mathbf{E}}\exp(i(kz-\omega t))\big)\ ,$$

show that the first equation is equivalent to

$$\tilde{\mathbf{E}} = E_0\big(\hat{x} + \hat{y}\, b e^{i\phi}\big)\ .$$

(**b**)
Sketch the locus of $\mathbf{E}/E_0$ in the x-y plane for the following cases:

$\phi = 0$, $b = 1$

$\phi = 0$, $b = 2$

$\phi = \pi/2$, $b = -1$

$\phi = \pi/4$, $b = 1$

(**c**)
In cases where E_x is nonzero, define the complex constant α according to

$$\begin{pmatrix}\tilde{E}_x\\ \tilde{E}_y\end{pmatrix} \propto \frac{1}{\sqrt{1+|\alpha|^2}}\begin{pmatrix}1\\ \alpha\end{pmatrix}\ .$$

The right-hand side of this equation (including the normalizing factor) is called the *Jones vector* J. Write the Jones vectors for the four waves described in (**b**).

2. Consider an ideal linear polarizer with its transmission axis at an arbitrary angle ϕ with respect to the x axis. It acts on a fully polarized plane EM wave that is characterized by initial and final Jones vectors J and J' (only J is normalized), such that

$$J' \equiv MJ\ ,$$

where M is a 2×2 *Jones matrix* representing the polarizer. Calculate M. (As usual, the beam direction is z, ϕ is an angle in the xy plane, and ϕ is positive as one rotates from $\hat{x}$ toward $\hat{y}$.)

3. A *wave plate* consists of a single crystal in which plane EM waves that are linearly polarized in the "slow" ("fast") direction propagate with phase velocity c/n_{slow} (c/n_{fast}), where $n_{\text{slow}} > n_{\text{fast}}$ due to lack of cubic symmetry in the crystal lattice. Most interesting is the *quarter-wave plate*, which has a thickness D such that

$$kD(n_{\text{slow}} - n_{\text{fast}}) = \pi/2\ .$$

Usually a quarter-wave plate is deployed with its slow axis bisecting the $\hat{x}$ and $\hat{y}$ axes, where $\hat{z}$ is the direction of wave propagation; the plate's fast axis is perpendicular to its slow axis.
(**a**)
Calculate the Jones matrix M that characterizes this quarter-wave plate (constant multiplicative phase factors are unimportant).
(**b**)
Starting with light that is unpolarized, *i.e.* in which there is no fixed phase relationship between $\tilde{E}_x$ and $\tilde{E}_y$, it is possible to obtain fully circularly polarized light using only a quarter-wave plate and a linear polarizer. Specify in which order these elements should be traversed by the beam. Using the Jones matrix that you calculated for the quarter-wave plate, prove that your design will work.

4. Using a combination of optical elements (linear polarizer or wave plate), design a system that will pass right-hand circularly polarized light without changing its polarization, but will completely block left-hand circularly polarized light. This system is called a *right-hand circular analyzer*. Use Jones matrices to prove that your design will work. (According to the usual convention, if you take a snapshot of a right-hand polarized EM wave, the electric field vector traces the thread pattern of a right-handed screw pointed along $\hat{z}$.)

5. Griffiths 9.31

6. Show that the characteristic impedance $Z_0 \equiv \Delta V/I$ of the coaxial cable in the previous problem is

$$Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}}\ln\frac{b}{a}$$

and that this result is equivalent to

$$Z_0 = \sqrt{\frac{L'}{C'}}$$

where L' and C' are the cable's inductance and capacitance per unit length, respectively.

7. Prove that

$$\sum_{n=1}^{N} \exp(i\phi_n) = \frac{\sin N\Delta\phi/2}{\sin\Delta\phi/2}\exp(i\bar{\phi}),$$

where

$$\Delta\phi \equiv \phi_{n+1} - \phi_n,$$

and $\bar{\phi}$ is the average of the ϕ_n.

8. The result of the previous problem can be used to calculate the diffraction pattern of an N-slit system.

Consider a very thin, perfectly conducting screen at $z = 0$, upon which a plane EM wave is incident from $z < 0$, linearly polarized along $\hat{x}$. Macroscopically, we know that the wave is fully reflected; nothing is transmitted. Microscopically, free electrons in the screen are set into vibration by the incident electric field. Taken individually, these dipoles would radiate in the usual electric dipole pattern. However, when the effects of all the dipoles are combined, along $+\hat{z}$ they radiate a plane wave that exactly cancels the incident wave; and along $-\hat{z}$ they radiate a plane reflected wave.

Now cut a narrow slit in the screen, along $z = 0, y =$ constant. This is equivalent to removing a line of dipoles – or to inserting a line of *oppositely vibrating* dipoles. Then for $z > 0$ the resulting diffraction pattern is merely the radiation pattern of the inserted line of oppositely vibrating dipoles (everything else cancels out). (This argument is due to *Babinet*.)

Consider N equally spaced slits of the type just described. Put the observer at $x = y = 0$, $z = \infty$; direct the incident beam at an angle $\psi = \arctan(k_y/k_z)$ to the z axis. The observer sees the radiation from N lines of dipoles – but the phase of each line of dipoles differs from that of the next line by $kb\sin\psi$, where b is the slit spacing and k is the wave vector's magnitude.

Calculate the diffraction pattern

$$S(\psi)/S(\psi = 0)\ ,$$

where S is the magnitude of the Poynting vector seen by the observer.

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Solution Set 11 (by Daniel Larson)

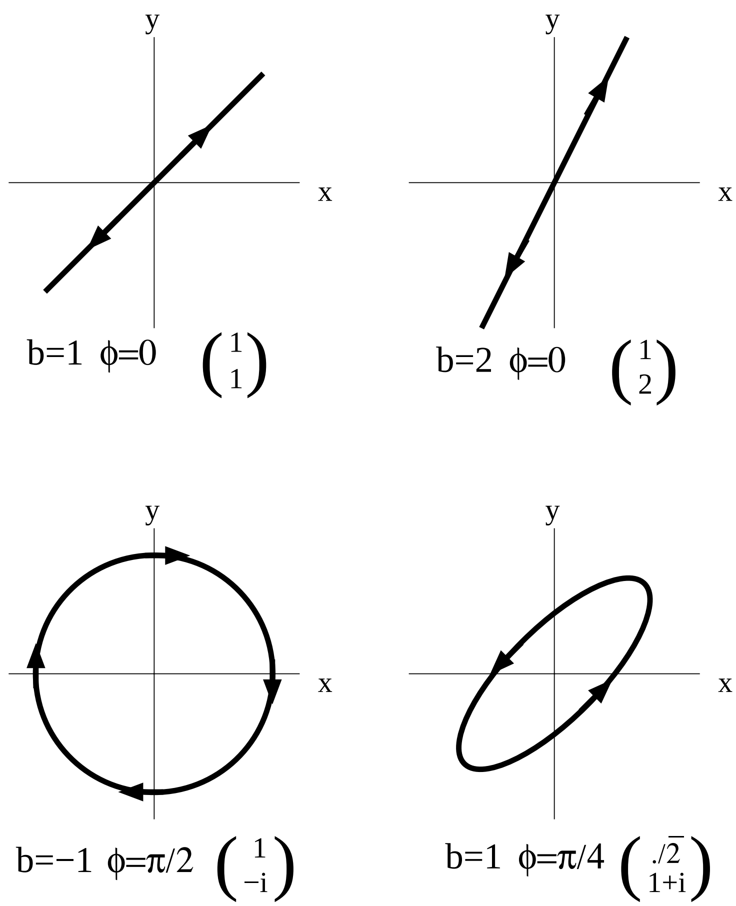
1. **Polarization**

(a) We want to convert the real electric field $\mathbf{E}$ into a complex electric field.

$$\mathbf{E} = E_0 \mathrm{Re}\Big[e^{i(kz-\omega t)}\Big]\,\hat{\mathbf{x}} + E_0 b \mathrm{Re}\Big[e^{i(kz-\omega t+\phi)}\Big]\,\hat{\mathbf{y}} = \mathrm{Re}\Big[E_0\big(\,\hat{\mathbf{x}} + \hat{\mathbf{y}} b e^{i\phi}\big)e^{i(kz-\omega t)}\Big]$$

$$\tilde{\mathbf{E}} = E_0(\,\hat{\mathbf{x}} + b e^{i\phi}\,\hat{\mathbf{y}}) = \begin{pmatrix} 1 \\ b e^{i\phi} \end{pmatrix}$$

(b) The real E-field vector lives in the xy-plane. We want to sketch the path traced by the tip of the E-field for *fixed* z as we let the time increase. The easiest way to go about determining the path is to set $z = 0$, and then use the values for b and ϕ in the real vector $\mathbf{E}$ and plot some sample points for $t = 0, \frac{\pi}{2\omega}, \frac{\pi}{\omega}, \ldots$.



These plots represent light with linear polarization (top two), right-hand circular polarization (lower left) and left-hand elliptical polarization (lower right).

(c) The Jones vectors are given by $\dfrac{1}{\sqrt{1+b^2}}\begin{pmatrix} 1 \\ b e^{i\phi} \end{pmatrix}$; the un-normalized vectors are shown next to the plots above. When properly normalized they are, respectively:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \frac{1}{\sqrt{2}}(1+i) \end{pmatrix}$$

2. **Polarizer** A Jones vector $\mathbf{J}$ represents the polarization of a beam of light. When that beam passes through a polarizer, the polarization is changed. We would like to be able to represent the effect of the polarizer by a 2×2 matrix that acts on $\mathbf{J}$ representing the initial beam and gives back $\mathbf{J}'$ representing the final beam: $M\mathbf{J} = \mathbf{J}'$. We want to find the matrix M for a polarizer aligned with its transmission axis at an arbitrary angle ϕ with respect to the x-axis.

   Let's start by solving a simpler system, where the transmission axis is along the x_A-axis. We're going to change basis later, so I want to label this $x-y$ coordinate system with the subscript A. All the vector coordinates and matrices given with respect to this basis will also have a subscript A. Incoming light with x-polarization is represented by the Jones vector $\mathbf{J}_A = \binom{1}{0}_A$, and it will pass through the polarizer unchanged. (It's traveling along the "transmission" axis, after all.) On the other hand, light with y-polarization, $\mathbf{J}_A = \binom{0}{1}_A$, will be completely blocked. These conditions are enough to specify the matrix M_A in the A-basis.

$$M_A \begin{pmatrix}1\\0\end{pmatrix}_A = \begin{pmatrix}1\\0\end{pmatrix}_A \qquad M_A \begin{pmatrix}0\\1\end{pmatrix}_A = \begin{pmatrix}0\\0\end{pmatrix}_A \qquad \Rightarrow M_A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

   Now we want to change basis to a new one (with coordinates x_B and y_B) such that the transmission axis of the polarizer (the x_A-axis) makes an angle of ϕ with the new x_B-axis. Let's let the matrix that accomplished this change of basis be called R, so that a Jones vector expressed in the B-basis, $\mathbf{J}_B$, will be expressed as the vector $\mathbf{J}_A = R\mathbf{J}_B$ in the A-basis. We can also go back the other way: $\mathbf{J}_B = R^{-1}\mathbf{J}_A$. What we really want is the matrix M_B, where $M_B\mathbf{J}_B = \mathbf{J}'_B$. In terms of M_A and R we find:

$$M_A\mathbf{J}_A = \mathbf{J}'_A \;\Rightarrow\; M_A R\mathbf{J}_B = R\mathbf{J}'_B \;\Rightarrow\; R^{-1}M_A R\mathbf{J}_B = \mathbf{J}'_B \;\Rightarrow\; M_B = R^{-1}M_A R$$

   So now all we need to do is determine R. Consider a unit vector on the x_B-axis. In the B-coordinates it is given by $\binom{1}{0}_B$ while in the A-coordinates it is given by $\binom{\cos\phi}{-\sin\phi}_A$. Thus $R\binom{1}{0}_B = \binom{\cos\phi}{-\sin\phi}_B$. You can do the same analysis starting with $\binom{0}{1}_B$. The matrix R is then $R = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$

$$M_B = R^{-1}M_A R = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_A\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{pmatrix}_B$$

3. **Wave Plate**

   (a) We can use the same method to find the matrix for the quarter-wave plate. Namely, first solve the problem in an easier situation, where the slow axis is along the x_A-axis, and then change to the B-coordinates where the slow axis is at 45° with respect to the x_B axis. We can use the same R matrix we found above, with $\phi = 45°$.

   So we first need to find M_A. If we have light polarized along the slow axis, $\mathbf{J}_A = \binom{1}{0}_A$, it will pass through the quarter-wave plate and pick up some arbitrary phase that we might as well take to be zero. Then when light polarized along the fast axis, $\mathbf{J}_A = \binom{0}{1}_A$, passes though, it picks up a quarter-wavelength *less* phase, so it is $\pi/2$ *behind* the wave along the slow axis. So if the slow-axis wave emerges with $\mathbf{J}'_A = \binom{1}{0}_A$, the fast-axis wave will emerge with Jones vector $\mathbf{J}'_A = \binom{0}{e^{-i\pi/2}}_A = \binom{0}{-i}$. This means the matrix for the quarter-wave plate is given by $M_A^{1/4} = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}_A$ in the A-coordinates. Using R and R^{-1} with $\phi = \pi/4$, we find:

$$M_B^{1/4} = R^{-1}M_A^{1/4}R = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}_A\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1-i & 1+i \\ 1+i & 1-i \end{pmatrix}_B$$

This might not be exactly what you expected, but remember we can multiply the whole matrix by an overall phase without changing anything. Choosing the phase $e^{i\pi/4} = \frac{1}{\sqrt{2}}(1+i)$ we find

$$M_B = e^{i\pi/4}\frac{1}{2}\begin{pmatrix} 1-i & 1+i \\ 1+i & 1-i \end{pmatrix}_B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}_B$$

(b) Now we can use these matrices to design optical systems. We want to take unpolarized light and make it circularly polarized. First notice that $M_B^{1/4}\binom{1}{0} = \frac{1}{\sqrt{2}}\binom{1}{i}$, namely the quarter wave plate turns x-polarized light into LH circularly polarized light. Thus by first using a linear polarizer aligned along the x-axis, we can turn the unpolarized beam into an x-polarized beam, which can then pass through the quarter wave plate and become LH circularly polarized. In terms of matrices, our system of x-polarizer then quarter-wave plate is represented by the matrix

$$M_{\text{tot}} = M^{1/4}M^x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ i & 0 \end{pmatrix}$$

Note that the order of matrices is "opposite" the order in which the optical elements are traversed (as reading from left to right). This is because it is the matrix on the right that is the first to act on any Jones vector. To test if this matrix does what we want, multiply it by any random vector. $M_{\text{tot}}\binom{a}{b} = \frac{1}{\sqrt{2}}\binom{a}{ai} = \frac{a}{\sqrt{2}}\binom{1}{i}$, which is indeed LH circularly polarized.

4. **Circular Analyzer** Now we want to make a RH circular analyzer that passes RH circularly polarized light unchanged and completely blocks LH circularly polarized light. The idea is to use a quarter-plate to make circularly polarized light into linearly polarized light, then use a linear polarizer to allow only the component you want through, and then use the quarter-wave plate again to make the linearly polarized light circular again. The important observation is that a quarter-wave plate turns x-polarized light into LH circularly polarized, LH circular into y-polarized, y-polarized into RH circular, and RH-circular into x-polarized. (This is easier to visualize if you draw a little flow chart.) So we will send a light beam through the quarter-wave plate. The RH components will turn into x-polarized light and the LH components will become y-polarized. Then we use a linear polarizer with the x-axis as its transmission axis, so only x-polarized light gets through. Then we can use 3 quarter-wave plates, or equivalently one quarter-wave plate rotated by pi, to convert the x-polarized light back to RH circular. In matrices this becomes:

$$(M^{1/4})^3M^xM^{1/4} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & i \\ i & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

Now to test it, multiply by a the vector $\mathbf{J} = \binom{a}{b}$, which gives (dropping the irrelevant overall minus sign) $\frac{1}{2}\binom{a+ib}{b-ia} = \frac{1}{2}(a+ib)\binom{1}{-i}$. Now, RH circularly polarized light has $a=1, b=-i$, so the output is $\binom{1}{-i}$ which is RH circular, just as we want. On the other hand, LH circular light has $a=1, b=i$ which yields 0, so it is completely blocked. This design works as desired!

5. **Griffiths 9.31**

(a) The first part of the problem consists of plugging equations (9.197) into (9.177) and (9.175). To take the derivatives in Maxwell's equations we need the formulas for divergence and curl in cylindrical coordinates, as found in the front cover.

$$\nabla\cdot\mathbf{E} = \nabla\cdot\left(\frac{A\cos(kz-\omega t)}{s}\,\hat{\mathbf{s}}\right) = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{A\cos(kz-\omega t)}{s}\right) = 0 \qquad \nabla\cdot\mathbf{B} = \frac{1}{s}\frac{\partial}{\partial\phi}(B_\phi) = 0$$

$$\nabla\times\mathbf{E} = \frac{\partial E_s}{\partial z}\,\hat{\boldsymbol{\phi}} - \frac{1}{s}\frac{\partial E_s}{\partial\phi}\,\hat{\mathbf{z}} = -\frac{1}{s}Ak\sin(kz-\omega t)\,\hat{\boldsymbol{\phi}} \quad \text{while} \quad -\frac{\partial\mathbf{B}}{\partial t} = -\frac{A\omega}{cs}\sin(kz-\omega t)\,\hat{\boldsymbol{\phi}}$$

The above two terms are equal because $\omega = ck$. Similarly,

$$\nabla \times \mathbf{B} = -\frac{\partial B_\phi}{\partial z}\,\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial}{\partial s}(sB_\phi)\,\hat{\mathbf{z}} = \frac{Ak}{cs}\sin(kz-\omega t)\,\hat{\mathbf{s}} \quad \text{while} \quad \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \frac{A\omega}{c^2 s}\sin(kz-\omega t)\,\hat{\mathbf{s}}$$

The boundary conditions are easily satisfied: $E^{\parallel} = E_z = 0$ and $B^{\perp} = B_s = 0$.

(b) To find the charge density, we can use Gauss's Law with a cylindrical surface with radius s and length dz enclosing the inner cylinder. $\oint \mathbf{E}\cdot d\mathbf{a} = \frac{A}{s}\cos(kz-\omega t)(2\pi s)dz = Q_{\text{enc}}/\epsilon_0 = \lambda dz/\epsilon_0 \Rightarrow \lambda = 2\pi\epsilon_0 A\cos(kz-\omega t)$.

To determine I we use an Amperian loop of radius s. We need to take into account both the "regular" current and also the displacement current. However, since the displacement current is $\epsilon_0\frac{\partial \mathbf{E}}{\partial t}$, and $\mathbf{E}$ is only radial, none of it will pass through a loop enclosing the inner cylinder. Thus $\oint \mathbf{B}\cdot d\mathbf{l} = \frac{A}{cs}\cos(kz-\omega t)(2\pi s) = \mu_0 I_{\text{enc}} \Rightarrow I = \frac{2\pi A}{\mu_0 c}\cos(kz-\omega t)$. Note that $I = \frac{\lambda}{\epsilon_0\mu_0 c}$.

6. **Characteristic Impedance** The characteristic impedance is defined as $Z_0 \equiv \Delta V/I$. In Problem 6 of Homework #2 we calculated the potential difference and capacitance per unit length for a system of coaxial cylinders. We found $\Delta V = \frac{\lambda}{2\pi\epsilon_0}\ln(b/a)$, and $C' = 2\pi\epsilon_0/\ln(b/a)$. So

$$Z_0 = \frac{\Delta V}{I} = \frac{\lambda \ln(\frac{b}{a})}{2\pi\epsilon_0}\frac{\epsilon_0\mu_0 c}{\lambda} = \frac{1}{2\pi}\frac{\mu_0}{\sqrt{\mu_0\epsilon_0}}\ln\left(\frac{b}{a}\right) = \frac{1}{2\pi}\sqrt{\frac{\mu_0}{\epsilon_0}}\ln\frac{b}{a}$$

The inductance per unit length for this arrangement was calculated in Example 7.13 in the text. $L' = \frac{\mu_0}{2\pi}\ln\left(\frac{b}{a}\right)$. Thus

$$\sqrt{\frac{L'}{C'}} = \sqrt{\frac{\mu_0}{2\pi}\ln\left(\frac{b}{a}\right)\frac{\ln\left(\frac{b}{a}\right)}{2\pi\epsilon_0}} = \frac{1}{2\pi}\sqrt{\frac{\mu_0}{\epsilon_0}}\ln\left(\frac{b}{a}\right) = Z_0$$

7. **Identity** The first important thing to notice is that since $\phi_n - \phi_{n-1} = \Delta\phi$ for all n, we can write each ϕ_n in terms of ϕ_1 and $\Delta\phi$: $\phi_n = \phi_1 + (n-1)\Delta\phi$. Expanding the sum and using this relation, we get:

$$\begin{aligned}\sum_{n=1}^{N} e^{i\phi_n} &= e^{i\phi_1} + e^{i\phi_2} + \cdots + e^{i\phi_N} = e^{i\phi_1} + e^{i(\phi_1+\Delta\phi)} + e^{i(\phi_1+2\Delta\phi)} + \cdots + e^{i(\phi_1+(N-1)\Delta\phi)} \\ &= e^{i\phi_1}\left(1 + e^{i\Delta\phi} + e^{2i\Delta\phi} + \cdots + e^{i(N-1)\Delta\phi}\right)\end{aligned}$$

The term in parentheses looks like a geometric series. Notice that $(1+a+a^2+\cdots+a^{N-1})(1-a) = 1-a^N \Rightarrow (1+a+a^2+\cdots+a^{N-1}) = (1-a^N)/(1-a)$. Using this identity with $a = e^{i\Delta\phi}$, we get

$$\begin{aligned}\sum_{n=1}^{N} e^{i\phi_n} &= e^{i\phi_1}\frac{1-e^{iN\Delta\phi}}{1-e^{i\Delta\phi}} = e^{i\phi_1}\frac{1-e^{iN\Delta\phi}}{1-e^{i\Delta\phi}}\frac{e^{-iN\Delta\phi/2}e^{iN\Delta\phi/2}}{e^{-i\Delta\phi/2}e^{i\Delta\phi/2}} = e^{i\phi_1}e^{iN\Delta\phi/2}e^{-i\Delta\phi/2}\frac{e^{-iN\Delta\phi/2}-e^{iN\Delta\phi/2}}{e^{-i\Delta\phi/2}-e^{i\Delta\phi/2}} \\ &= e^{i(\phi_1+(N-1)\Delta\phi/2)}\frac{\sin(N\Delta\phi/2)}{\sin(\Delta\phi/2)}\end{aligned}$$

We're getting close. Let's step aside and calculate $\bar{\phi}$.

$$\bar{\phi} = \frac{1}{N}\sum_{n=1}^{N}\phi_n = \frac{1}{N}\sum_{n=1}^{N}(\phi_1+(n-1)\Delta\phi) = \frac{1}{N}\left(N\phi_1 + \Delta\phi\sum_{n=1}^{N}N(n-1)\right) = \phi_1+\frac{1}{N}\Delta\phi\frac{N(N-1)}{2} = \phi_1+(N-1)\frac{\Delta\phi}{2}$$

How nice! This is exactly what is in the exponent of the phase factor above. Thus

$$\sum_{n=1}^{N} e^{i\phi_n} = \frac{\sin\left(N\frac{\Delta\phi}{2}\right)}{\sin\left(\frac{\Delta\phi}{2}\right)}e^{i\bar{\phi}}$$

8. **Diffraction** We have a screen with N identical long thin slits cut in it. First let's deal with the situation when $\psi = 0$; i.e. the incoming wave is normal to the screen. Let $\mathbf{E}_0$ be the electric field at the location of the observer due to one of the slits. Since the slits are identical, and the light from each of them is in phase with the others, each one contributes $\mathbf{E}_0$ to the total E-field seen by the observer. Thus $\mathbf{E}_{\text{tot}} = \mathbf{E}_0 + \mathbf{E}_0 + \cdots + \mathbf{E}_0 = N\mathbf{E}_0$. The magnitude of the Poynting vector measured by the observer is $S = \left|\frac{1}{\mu_0 c}\mathbf{E}_{\text{tot}}\right|^2 = \frac{1}{\mu_0 c}N^2|\mathbf{E}_0|^2$.

   Now consider the case when the incident wave hitting the screen with the slits is at an angle ψ. This means that the electric field from each of the slits will be out of phase from the one below it by $\Delta\phi = kb\sin\psi$. Let us define the overall phase so that the E-field from the first slit is $\mathbf{E}_0 e^{i\phi_1}$. Then

$$\mathbf{E}_{\text{tot}} = \mathbf{E}_1 + \mathbf{E}_2 + \cdots = \mathbf{E}_0 e^{i\phi_1} + \mathbf{E}_0 e^{i\phi_2} + \cdots + \mathbf{E}_0 e^{i\phi_N} = \mathbf{E}_0 \sum_{n=1}^{N} e^{i\phi_n} = \mathbf{E}_0 e^{i\bar{\phi}} \frac{\sin\left(N\frac{\Delta\phi}{2}\right)}{\sin\left(\frac{\Delta\phi}{2}\right)}$$

   Now we can compute S.

$$S(\psi) = \frac{1}{\mu_0 c}|\mathbf{E}_{\text{tot}}|^2 = \frac{1}{\mu_0 c}|\mathbf{E}_0|^2 \frac{\sin^2\left(N\frac{\Delta\phi}{2}\right)}{\sin^2\left(\frac{\Delta\phi}{2}\right)}$$

   (The complex exponential has a modulus of one: $|e^{i\bar{\phi}}|^2 = 1$.) Now when we make the ratio $S(\psi)/S(0)$, the constants and $|\mathbf{E}_0|^2$ pieces cancel, leaving just an N^2 from the denominator. Using $\Delta\phi = kb\sin\psi$ we find:

$$\frac{S(\psi)}{S(\psi = 0)} = \frac{\sin^2\left(\frac{1}{2}Nkb\sin\psi\right)}{N^2\sin^2\left(\frac{1}{2}kb\sin\psi\right)}$$

   If we evaluate this result for $\psi = 0$, using the small angle approximation on the outer sine functions, we get $S(0)/S(0) = 1$, as it must.

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Physics 110A Fall 2001   Section 1 (*Strovink*)

MIDTERM EXAMINATION 1

Directions: Do all 3 problems, which have unequal weight. This is a closed-book closed-note exam except for one $8\frac{1}{2} \times 11$ inch sheet containing any information you wish on both sides. A photocopy of the four inside covers of Griffiths is included with the exam. Calculators are not needed, but you may use one if you wish. Laptops and palmtops should be turned off. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Express your answer in terms of the quantities specified in the problem. Box or circle your answer.

Problem 1. (45 points)
A surface charge of uniform density σ_0 Coul/m^2 is glued onto a spherical shell of radius R that is centered at the origin.

(**a**) (10 points)
Relative to ∞, find the potential V_0 at the origin.

(**b**) (5 points)
How much work W was done to move the charge from ∞ to the shell?

(**c**) (10 points)
The shell is now split along its "equator" into two hemispheres, and the south hemisphere is thrown away. Find the new potential $V_{1/2}$ at the origin.

(**d**) (20 points)
For the conditions of part (**c**), calculate the potential V_N at the "north pole" $(0, 0, R)$.

Problem 2. (25 points)
A point charge q is held at a distance z above an infinite conducting plane that is grounded ($V = 0$). Calculate the surface charge density σ_s on the plane at a distance $s \gg z$ from the charge. Accuracy to lowest nonvanishing order in z/s is sufficient.

Problem 3. (30 points)
A thin phonograph record is composed of a material that has a uniform volume charge density; the total charge is Q. The record has radius R and rotates on a turntable at angular velocity $\vec{\omega}$. Calculate the magnetic field at the center of the record.

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SOLUTION TO MIDTERM EXAMINATION 1

Directions: Do all 3 problems, which have unequal weight. This is a closed-book closed-note exam except for one $8\frac{1}{2} \times 11$ inch sheet containing any information you wish on both sides. A photocopy of the four inside covers of Griffiths is included with the exam. Calculators are not needed, but you may use one if you wish. Laptops and palmtops should be turned off. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Express your answer in terms of the quantities specified in the problem. Box or circle your answer.

Problem 1. (45 points)
A surface charge of uniform density σ_0 Coul/m^2 is glued onto a spherical shell of radius R that is centered at the origin.

(**a**) (10 points)
Relative to ∞, find the potential V_0 at the origin.

Solution:
This part of the problem is spherically symmetric. Outside the shell, the potential is that of a point charge. Inside the shell, there is no charge, so the potential there is the same as at the shell's surface. Therefore

$$\begin{aligned} 4\pi\epsilon_0 V_0 &= \frac{q}{R} \\ &= \frac{4\pi R^2 \sigma_0}{R} \\ V_0 &= \frac{\sigma_0 R}{\epsilon_0} \;. \end{aligned}$$

(**b**) (5 points)
How much work W was done to move the charge from ∞ to the shell?

Solution:

$$\begin{aligned} W &= \tfrac{1}{2} \int d\tau' \rho(\mathbf{r}') V(\mathbf{r}') \\ &= \tfrac{1}{2} q V(R) \\ &= \tfrac{1}{2} 4\pi R^2 \sigma_0 \frac{\sigma_0 R}{\epsilon_0} \\ W &= \frac{2\pi R^3 \sigma_0^2}{\epsilon_0} \;. \end{aligned}$$

(**c**) (10 points)
The shell is now split along its "equator" into two hemispheres, and the south hemisphere is thrown away. Find the new potential $V_{1/2}$ at the origin.

Solution:
We could have obtained the answer to (**a**) by doing the integral

$$4\pi\epsilon_0 V_0 = \int d\tau' \frac{\rho(\mathbf{r}')}{r'} \;.$$

Now, with half of the shell removed, the integral is half as big. Therefore

$$V_{1/2} = \frac{V_0}{2} = \frac{\sigma_0 R}{2\epsilon_0} \;.$$

(**d**) (20 points)
For the conditions of part (**c**), calculate the potential V_N at the "north pole" $(0, 0, R)$.

Solution:
Now we need actually to do an integral. Consider a ring $d\theta'$ of charge, where θ' is the angle measured from the north pole. This ring has area $da' = 2\pi R^2 \sin\theta' d\theta'$ and is located a distance $r' = 2R \sin\frac{\theta'}{2}$ from the north pole. The contribution from this ring to the potential at the north pole is

$$\begin{aligned} 4\pi\epsilon_0 \, dV_N &= \frac{\sigma_0 da'}{r'} \\ &= \frac{\sigma_0 2\pi R^2 \sin\theta'}{2R \sin\frac{\theta}{2}} d\theta' \\ &= \frac{\pi R \sigma_0 \sin\theta'}{\sin\frac{\theta'}{2}} d\theta' \;. \end{aligned}$$

Substituting

$$\sin\theta' = \sin\left(\tfrac{\theta'}{2} + \tfrac{\theta'}{2}\right)$$
$$= 2\sin\tfrac{\theta'}{2}\cos\tfrac{\theta'}{2}\ ,$$

we have

$$4\pi\epsilon_0\, dV_N = \frac{\pi R\sigma_0 2\sin\frac{\theta'}{2}\cos\frac{\theta'}{2}}{\sin\frac{\theta'}{2}}d\theta'$$
$$= 2\pi R\sigma_0\cos\tfrac{\theta'}{2}2d\tfrac{\theta'}{2}\ .$$

Integrating over $0 < \theta' < \frac{\pi}{2}$,

$$4\pi\epsilon_0 V_N = 4\pi R\sigma_0\int_0^{\pi/4}\cos\tfrac{\theta}{2}d\tfrac{\theta}{2}$$
$$= 4\pi R\sigma_0\sin\tfrac{\pi}{4}$$
$$V_N = \frac{\sigma_0 R}{\sqrt{2}\epsilon_0}\ .$$

As a check, if we had integrated θ' all the way to π, including both hemispheres, we would have recovered the answer to (**a**).

Problem 2. (25 points)
A point charge q is held at a distance z above an infinite conducting plane that is grounded ($V = 0$). Calculate the surface charge density σ_s on the plane at a distance $s \gg z$ from the charge. Accuracy to lowest nonvanishing order in z/s is sufficient.

Solution:
For $z > 0$, the effect of the charge that is induced on the conducting plane is the same as that of an image charge $-q$ a distance z below the plane. Together the physical charge and the image charge form a physical dipole with moment $\mathbf{p} = \hat{z}q2z$. At a cylindrical radius $s \gg z$, the field of the physical dipole is approximately the same as that of an ideal dipole:

$$\frac{4\pi\epsilon_0 r^3}{p}\mathbf{E} = 3\hat{r}(\hat{r}\cdot\hat{p}) - \hat{p}$$
$$= 3\hat{s}(\hat{s}\cdot\hat{z} = 0) - \hat{z}$$
$$4\pi\epsilon_0\mathbf{E} = -\hat{z}\frac{2qz}{s^3}\ .$$

The surface charge density on the conductor is just $\epsilon_0 E_z$, so

$$\sigma_s = -\frac{qz}{2\pi s^3}\ .$$

Apart from factors of order unity, the answer $-qz/s^3$ could be guessed. Since a dipole is involved, the result must be proportional to its moment and thus to z. Given that, $-qz/s^3$ is the only acceptable combination of the available variables that has the dimensions of a surface charge density. This argument is worth some part credit.

This problem could also be approached by considering separately the electric fields from the physical and image charges, expanding them in powers of z/s, and retaining the leading terms that do not cancel. If you attempted to do this and fouled it up, you shouldn't expect excessive part credit, as such an approach doesn't require excessive physical insight.

Problem 3. (30 points)
A thin phonograph record is composed of a material that has a uniform volume charge density; the total charge is Q. The record has radius R and rotates on a turntable at angular velocity $\vec{\omega}$. Calculate the magnetic field at the center of the record.

Solution:
Again we need to do an integral. Define $\hat{z} \equiv \hat{\omega}$ and s to be the (cylindrical) radius. Consider an element ds of the record, located a distance s from its center. The charge dQ on this element is

$$
\begin{aligned}
dQ &= Q\frac{2\pi s\, ds}{\pi R^2} \\
&= \frac{2Qs}{R^2} ds \ .
\end{aligned}
$$

This charge rotates once every $2\pi/\omega$ seconds, so the element carries a current

$$
\begin{aligned}
dI &= \frac{\omega}{2\pi}\frac{2Qs}{R^2} ds \\
&= \frac{\omega Q s}{\pi R^2} ds \ .
\end{aligned}
$$

When one applies the Biot-Savart law, one finds that a circular loop of current I_0 and radius s_0 has a central field equal to $\mu_0 I_0/2s_0$. Therefore the contribution of the record element ds to the central magnetic field is

$$
\begin{aligned}
d\mathbf{B} &= \hat{z}\frac{\mu_0}{2s} dI \\
&= \hat{z}\frac{\mu_0}{2s}\frac{\omega Q s}{\pi R^2} ds \\
&= \hat{z}\frac{\mu_0 \omega Q}{2\pi R^2} ds \\
\mathbf{B} &= \hat{z}\frac{\mu_0 \omega Q}{2\pi R^2}\int_0^R ds \\
\mathbf{B} &= \hat{\omega}\frac{\mu_0 \omega Q}{2\pi R} \ .
\end{aligned}
$$

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MIDTERM EXAMINATION 2

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Problem 1. (50 points)
A cylindrically symmetric region is bounded by $-\infty < z < \infty$ and $s < s_0$ (s is the cylindrical radius in Griffiths' notation). Within this region, the magnetic field may be obtained from the vector potential

$$\mathbf{A}(s) = \hat{z}\mu_0 C s^2 \ ,$$

where C is uniform, *i.e.* independent of $\mathbf{r}$. (You don't need to choose a particular gauge in order to work this problem, but, if it is helpful, you may work in Lorentz gauge $\nabla\cdot\mathbf{A} + \epsilon_0\mu_0\partial V/\partial t = 0$.)

(**a**) (15 points)
For this part, take C to be a (positive) constant, *i.e.* independent of time t as well as $\mathbf{r}$. Calculate the current density $\mathbf{J}$, flowing within this region, that produces $\mathbf{A}$. The *direction* and *sign* of your answer are important. (In this application, note that

$$\tfrac{4\pi}{\mu_0}\mathbf{A}(\mathbf{r}) \neq \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d\tau' \ ,$$

because the current-carrying region is infinite in extent.)

(**b**) (20 points)
For this part, take C to be a decaying function of time, *i.e.*

$$C(t) = C_0 \exp\left(-t/\tau\right) \ ,$$

where C_0 and τ are positive constants. Consider a rectangular loop drawn at constant azimuth ϕ, bounded by $0 < z < z_0$ and $0 < s < s_0$. Calculate the EMF $\mathcal{E}$ around this loop (the sign of your answer won't be graded).

(**c**) (15 points)
If you were asked to calculate the current density $\mathbf{J}$ for the conditions of part (**b**), where $\mathbf{A}$ decays with time, would you expect $\mathbf{J}$ to have the same dependence on s within our cylindrical region that you obtained in part (**a**)? Why or why not?

Problem 2. (50 points)
A nickel (five-cent coin) of radius a and thickness $d \ll a$ carries a uniform permanent magnetization

$$\mathbf{M} = \hat{z}M_0 \ ,$$

where M_0 is a positive constant and $\hat{z}$ is the nickel's axis of cylindrical symmetry.

(**a**) (30 points)
Calculate the magnetic field $\mathbf{B}(0,0,0)$ at the center of the nickel. The *direction* of $\mathbf{B}$ is important; express $\mathbf{B}$ to lowest nonvanishing order in d/a.

(**b**) (20 points)
In the plane $z = 0$, draw counterclockwise a large circular loop $s = b \gg a$ that is centered on the nickel. What magnetic flux Φ flows through this loop? The *sign* of Φ is important; express Φ to lowest nonvanishing order in d/b.

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SOLUTION TO MIDTERM EXAMINATION 2

Directions: Do both problems, which have equal weight. This is a closed-book closed-note exam except for two $8\frac{1}{2} \times 11$ inch sheets containing any information you wish on both sides. A photocopy of the four inside covers of Griffiths is included with the exam. Calculators are not needed, but you may use one if you wish. Laptops and palmtops should be turned off. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Express your answer in terms of the quantities specified in the problem. Box or circle your answer.

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$$\mathbf{A}(s) = \hat{z}\mu_0 C s^2 ,$$

where C is uniform, *i.e.* independent of $\mathbf{r}$. (You don't need to choose a particular gauge in order to work this problem, but, if it is helpful, you may work in Lorentz gauge $\nabla\cdot\mathbf{A} + \epsilon_0\mu_0\partial V/\partial t = 0$.)
(**a**) (15 points)
For this part, take C to be a (positive) constant, *i.e.* independent of time t as well as $\mathbf{r}$. Calculate the current density $\mathbf{J}$, flowing within this region, that produces $\mathbf{A}$. The *direction* and *sign* of your answer are important. (In this application, note that

$$\tfrac{4\pi}{\mu_0}\mathbf{A}(\mathbf{r}) \neq \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d\tau' ,$$

because the current-carrying region is infinite in extent.)
Solution:
Combining Ampère's law with Griffiths' vector identity (11),

$$\begin{aligned}\mu_0\mathbf{J} &= \nabla\times\mathbf{B} \\ &= \nabla\times(\nabla\times\mathbf{A}) \\ &= \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A} \\ &= 0 - \frac{1}{s}\frac{\partial}{\partial s}s\frac{\partial}{\partial s}\hat{z}\mu_0 C s^2 \\ \mathbf{J} &= -\hat{z}\,C\frac{1}{s}\frac{\partial}{\partial s}2s^2 \\ &= -\hat{z}\,4C .\end{aligned}$$

Notice that $\mathbf{A}$ and $\mathbf{J}$ point in opposite directions!
The above is the most direct path to the result. Alternatively, one may first evaluate $\mathbf{B}$:

$$\begin{aligned}\mathbf{B} &= \nabla\times\mathbf{A} \\ &= -\hat{\phi}\frac{\partial A_z}{\partial s} \\ &= -\hat{\phi}\,2\mu_0 C s ,\end{aligned}$$

where the term in the middle equation includes the only nonvanishing derivative in the curl. Then

$$\begin{aligned}\mu_0\mathbf{J} &= \nabla\times\mathbf{B} \\ &= \frac{\hat{z}}{s}\frac{\partial}{\partial s}sB_\phi \\ &= -\frac{\hat{z}}{s}\frac{\partial}{\partial s}s2\mu_0 C s \\ \mathbf{J} &= -\hat{z}\,4C .\end{aligned}$$

(**b**) (20 points)
For this part, take C to be a decaying function of time, *i.e.*

$$C(t) = C_0\exp(-t/\tau) ,$$

where C_0 and τ are positive constants. Consider a rectangular loop drawn at constant azimuth ϕ, bounded by $0 < z < z_0$ and $0 < s < s_0$. Calculate the EMF $\mathcal{E}$ around this loop (the sign of your answer won't be graded).
Solution:
The electric field is easily calculated from

$$\mathbf{E} = -\nabla V - \frac{\partial\mathbf{A}}{\partial t} .$$

The potential term integrates to zero around the loop and thus plays no role. Because **A** is in the $\hat{z}$ direction and vanishes on the z axis, the only contribution to the integral comes from the outer segment where $s = s_0$ and $d\mathbf{l} = \hat{z}dz$. Proceeding counterclockwise around the loop,

$$\begin{aligned}\mathcal{E} &= \oint \mathbf{E} \cdot d\mathbf{l} \\ &= -\oint \frac{\partial \mathbf{A}}{\partial t} \cdot d\mathbf{l} \\ &= -\int_{z_0}^{0} \frac{\partial}{\partial t} \hat{z}\mu_0 C_0 s_0^2 \exp\left(-t/\tau\right) \cdot \hat{z}dz \\ &= \int_{z_0}^{0} \frac{\mu_0 C_0 s_0^2 \exp\left(-t/\tau\right)}{\tau} dz \\ &= -\frac{\mu_0 C_0 s_0^2 z_0 \exp\left(-t/\tau\right)}{\tau} \ .\end{aligned}$$

The above is the most direct path to the result. Alternatively, one may first calculate the magnetic flux Φ through the loop, then obtain $\mathcal{E}$ from its time derivative. This flux is most easily evaluated by performing the line integral of **A** around the loop. Again proceeding counterclockwise,

$$\begin{aligned}\Phi &= \oint \mathbf{A} \cdot d\mathbf{l} \\ &= \int_{z_0}^{0} \mu_0 C_0 s_0^2 \exp\left(-t/\tau\right) dz \\ &= -\mu_0 s_0^2 z_0 C_0 \exp\left(-t/\tau\right) \ .\end{aligned}$$

This same flux may also be obtained by integrating **B** from part (**a**). Proceeding counterclockwise around the loop, $d\mathbf{a}$ is in the $\hat{\phi}$ direction, opposite to the direction of **B**. Therefore the flux is negative. Performing the integration,

$$\begin{aligned}\mathbf{B} &= -\hat{\phi}\, 2\mu_0 C s \\ \Phi &= \int \mathbf{B} \cdot d\mathbf{a} \\ &= -\int_0^{s_0} ds \int_0^{z_0} dz\, 2\mu_0 C s \\ &= -\mu_0 s_0^2 z_0 C \\ &= -\mu_0 s_0^2 z_0 C_0 \exp\left(-t/\tau\right) \ .\end{aligned}$$

With the same flux calculated either way, Faraday's law yields the EMF:

$$\begin{aligned}\mathcal{E} &= -\frac{d\Phi}{dt} \\ &= -\frac{\mu_0 C_0 s_0^2 z_0 \exp\left(-t/\tau\right)}{\tau} \ .\end{aligned}$$

(**c**) (15 points)
If you were asked to calculate the current density **J** for the conditions of part (**b**), where **A** decays with time, would you expect **J** to have the same dependence on s within our cylindrical region that you obtained in part (**a**)? Why or why not?
Solution:
Now that conditions are not static, Maxwell's corrected version of Ampère's Law is needed:

$$\mu_0 \mathbf{J} = \nabla \times \mathbf{B} - \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \ .$$

Though $\nabla \times \mathbf{B}$ has no s-dependence within our cylindrical region, the contribution of $d\mathbf{E}/dt$ to **J** is proportional to s^2, as is **A** itself. Therefore the Maxwell-corrected **J** will not have the same s-dependence as in part (**a**).

Problem 2. (50 points)
A nickel (five-cent coin) of radius a and thickness $d \ll a$ carries a uniform permanent magnetization

$$\mathbf{M} = \hat{z} M_0 \ ,$$

where M_0 is a positive constant and $\hat{z}$ is the nickel's axis of cylindrical symmetry.
(**a**) (30 points)
Calculate the magnetic field $\mathbf{B}(0, 0, 0)$ at the center of the nickel. The *direction* of **B** is important; express **B** to lowest nonvanishing order in d/a.
Solution:
The volume magnetization **M** yields a bound surface current $\mathbf{K}_b = \mathbf{M} \times \hat{n}$. Therefore $\mathbf{K}_b$ vanishes on the nickel's flat surfaces, and is equal to $\hat{\phi} M_0$ on its curved surface. A surface current on this thin curved strip $d \ll a$ is equivalent to a line current $I_b = K_b d$. Therefore **B** at the center is the same as the field from a circular loop.

Applying the Biot-Savart law,

$$\begin{aligned}\frac{4\pi}{\mu_0 I} d\mathbf{B}(\mathbf{r}=0) &= \frac{d\mathbf{l}' \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \\ &= \frac{\hat{\phi} s d\phi \times (-\hat{s})}{s^2} \\ &= \hat{z}\frac{d\phi}{a} \\ \mathbf{B}(0) &= \hat{z}\frac{\mu_0 K_b d}{2a} \\ &= \hat{z}\,\mu_0 M_0 \frac{d}{2a} \ .\end{aligned}$$

(**b**) (20 points)
In the plane $z=0$, draw counterclockwise a large circular loop $s=b \gg a$ that is centered on the nickel. What magnetic flux Φ flows through this loop? The *sign* of Φ is important; express Φ to lowest nonvanishing order in d/b.

Solution:
Far from the nickel, the field is that of a magnetic dipole with moment

$$\mathbf{m} = \hat{z}\, M_0 \pi a^2 d \ .$$

But the perfect-dipole approximation breaks down when we get close to the nickel, so it's tough to calculate Φ by integrating $\mathbf{B}$ over the loop's inner area.
The most straightforward approach uses the fact that the flux Φ through a loop is the integral of $\mathbf{A}$ around the loop; the dipole approximation for $\mathbf{A}$ will work well at the boundary of the loop, where $b \gg a$. First calculate $\mathbf{A}$:

$$\begin{aligned}\frac{4\pi}{\mu_0}\mathbf{A} &= \frac{\mathbf{m}\times\hat{r}}{r^2} \\ &= \frac{M_0\pi a^2 d}{r^2}\hat{z}\times(\hat{z}\cos\theta + \hat{s}\sin\theta) \\ &= \frac{M_0\pi a^2 d}{b^2}\hat{z}\times\hat{s} \\ &= \hat{\phi}\frac{M_0\pi a^2 d}{b^2} \\ \mathbf{A} &= \hat{\phi}\frac{\mu_0 M_0 a^2 d}{4b^2} \ .\end{aligned}$$

Since $\mathbf{A}$ is in the azimuthal direction, its line integral around the large circle is just $2\pi b A$, so

$$\Phi = \mu_0 \pi a^2 M_0 \frac{d}{2b} \ .$$

Note that, as $b \to \infty$, all the flux through the nickel is returned within the large circle, so $\Phi \to 0$.
The above is the most direct path to the result. An alternative approach starts from the equation

$$\oint \mathbf{B}\cdot d\mathbf{a} = 0 \ .$$

Choose a closed surface consisting of the plane $z=0$ plus the hemispherical cap $r=\infty$. The cap makes no contribution to the integral because $\mathbf{B}$ from a dipole diminishes as r^{-3}. The plane can be divided into $s<b$ and $s>b$. Since the surface integral over the plane vanishes, the inner and outer portions give equal and oppposite contributions. We evaluate the outer portion because the dipole approximation works well in that region.

$$\begin{aligned}\Phi &= \int_0^b ds \int_0^{2\pi} s\, d\phi\, B_z \\ &= -\int_b^\infty ds \int_0^{2\pi} s\, d\phi\, B_z \ .\end{aligned}$$

In the plane $z=0$, with $\hat{m}=\hat{z}$, the dipole's magnetic field is

$$\begin{aligned}\frac{4\pi r^3}{\mu_0 m}\mathbf{B} &= 3(\hat{m}\cdot\hat{r})\hat{r} - \hat{m} \\ \frac{4\pi s^3}{\mu_0 m}\mathbf{B} &= 3(\hat{z}\cdot\hat{s})\hat{s} - \hat{z} \\ &= -\hat{z} \\ \mathbf{B} &= -\hat{z}\frac{\mu_0 m}{4\pi s^3} \\ &= -\hat{z}\frac{\mu_0 \pi a^2 M_0 d}{4\pi s^3} \\ &= -\hat{z}\frac{\mu_0 a^2 M_0 d}{4 s^3} \ .\end{aligned}$$

Performing the integral over the outer region,

$$\begin{aligned}\Phi &= -\int_b^\infty ds \int_0^{2\pi} s\, d\phi \Big(-\frac{\mu_0 a^2 M_0 d}{4s^3}\Big) \\ &= \frac{\mu_0 a^2 M_0 d}{4} 2\pi \int_b^\infty \frac{ds}{s^2} \\ &= \mu_0 \pi a^2 M_0 \frac{d}{2b} \ .\end{aligned}$$

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FINAL EXAMINATION

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Problem 1. (30 points)
A conducting sphere of radius a, centered at the origin, carries a constant total charge q_0. Outside it, between radii $r = a$ and $r = b$, lies a spherical shell composed of an insulating dielectric with "frozen-in" polarization

$$\mathbf{P} = \hat{r}\frac{q_0}{4\pi r^2} ,$$

where q_0 is the same constant.

Calculate the electric field $\mathbf{E}(r)$ over the entire region $0 < r < \infty$. (Note that the dielectric constant ϵ is not defined or supplied here, and should not appear in your answer.)

Problem 2. (30 points). **Static fields**.
Please write down the following simple static electromagnetic fields in vacuum. Credit will be based entirely upon your answer; to receive any credit, your answer must be exactly correct, including the field's direction as well as its magnitude.

(**a**) (5 points)
$\mathbf{E}$ inside a parallel plate capacitor that holds charges $\pm q$ on plates of area a located at $y = \pm\frac{d}{2}$. (Note that the positive plate is on top.)

(**b**) (5 points)
$\mathbf{E}$ a distance r from the origin, where r is outside a spherically symmetric distribution of total charge Q that is centered at the origin.

(**c**) (5 points)
$\mathbf{E}$ at $(x \neq 0, 0, 0)$ produced by an ideal electric dipole of moment $\mathbf{p} = p_0\hat{z}$ that is centered at the origin.

(**d**) (5 points)
$\mathbf{B}$ a distance s outside a long thin wire carrying current I along $\hat{z}$.

(**e**) (5 points)
$\mathbf{B}$ at the center $(0, 0, 0)$ of a circular wire loop of radius b lying in the xy plane, carrying counterclockwise current I.

(**f**) (5 points)
$\mathbf{A}$ inside a long circular cylinder with its axis along $\hat{z}$, containing a magnetic field $\mathbf{B} = B_0\hat{z}$ inside, and $\mathbf{B} = 0$ outside.

Problem 3. (30 points)
A long thick cylindrical wire of radius b carries a steady current I_0 along its axis $\hat{z}$, uniformly distributed over the wire's cross section. At $t = 0$ the wire is cut with a thin saw to produce a thin gap in the region $-\frac{d}{2} < z < \frac{d}{2}$, with $d \ll b$. Neglect fringing effects near $s = b$.

(**a**) (15 points)
For a period of time after $t = 0$, the power supply that is connected to the distant ends of the wire forces the same current I_0 to continue to flow in the wire. Calculate the magnitude and direction of the magnetic field in the gap.

(**b**) (15 points)
At some later time, a resistor is substituted for the power supply, and the charge that accumulated on the faces $z = \pm\frac{d}{2}$ is allowed to drain away. While this charge is draining, would you expect the electric field in the gap to continue to be exactly uniform (independent of s)? Why or why not?

Problem 4. (40 points)
A nonrelativistic electron of mass m and charge e moves in vacuum in the xy plane under the influence of a constant uniform magnetic field B that is directed along the z axis. Because no other externally applied fields or mechanical forces exist, the electron travels very nearly in a periodic orbit. After one revolution, it is observed that the electron's kinetic energy has diminished slightly, by a factor $1 - \eta$, where $\eta \ll 1$.

In terms of B and fundamental constants, calculate η.

Problem 5. (30 points)
A plane wave of wavelength λ is normally incident on a system of thin slits at constant y in the aperture plane $z = 0$. An observer at $z = \infty$ observes the Fraunhofer-diffracted beam at a small angle $\theta \equiv \arctan(dy/dz)$. In each part of this problem, you are asked to calculate the diffraction-pattern ratio

$$R(\theta) \equiv \frac{I(\theta)}{I(0)} ,$$

where the intensity I is proportional to the square of the wave amplitude, *i.e.* to the time-averaged Poynting vector.
(**a**) (10 points)
Write down $R(\theta)$ for two thin slits at $y = \pm a/2$.
(**b**) (10 points)
Write down $R(\theta)$ for four thin slits, two at

$$y = +(a \pm b)/2 ,$$

and two at

$$y = -(a \pm b)/2 ,$$

where $a > b > 0$.
(**c**) (10 points)
Take the incident beam to be circularly polarized. Repeat part (**b**) under the same conditions, except that an $\hat{x}$ polarizer is placed behind the top pair of slits, and an otherwise identical $\hat{y}$ polarizer is placed behind the bottom pair.

Problem 6. (40 points)
A waveguide consists of an evacuated rectangular pipe that runs parallel to the $\hat{z}$ axis. The pipe has three perfectly conducting metal sides, at $x = 0$, $x = a$, and $y = 0$. These three sides are connected together in a "U" shape. The fourth (top) side, at $y = b$, is made of the same material but is insulated from the "U".

Operating in the TEM mode, the waveguide carries an electromagnetic pulse that travels in the $+\hat{z}$ direction. At $z = 0$ and $t = 0$, a snapshot is taken of the (nonzero) magnetic field $\mathbf{B}(x, y, z = 0, t = 0)$. Calculate $\mathbf{B}(x, y, z = 0, t = 0)$ within a multiplicative constant.