[1.] **Problem 7.2 from Griffiths**

A capacitor (capacitance, $C$) is charged to potential $V_0$. It is then connected to a resistor of resistance $R$ and at time $t = 0$ it begins discharging. Reference figures 7.5 (a) and (b) throughout the relevant parts of this problem.

(a) For the capacitor, find its charge, $Q(t)$. For the resistor, find its current, $I(t)$.

(b) Use equation 2.55 in Griffiths to write the original energy stored in the capacitor. Integrate equation 7.7 to show that the energy dissipated in the resistor is equivalent to the energy lost by the capacitor.

(c) Imagine a new scenario in which the capacitor is charging up from zero initial charge. Consider the circuit of figure 7.5 (b) in which a battery of constant voltage $V_0$ is connected at time $t = 0$. Find the $Q(t)$ and $I(t)$ as in part (a).

(d) Use $\int V_o I dt$ to calculate the total energy output by the battery. Calculate the energy dissipated in the resistor. Find the final stored energy of the capacitor. Determine the fraction of battery output energy that goes into the stored energy of the capacitor.

**Solution**

(a) The charge on any capacitor is given by $Q = CV$, where $C$ is the capacitance and $V$ is the potential across it (Griffiths covers capacitors in section 2.5.4). In this problem the charge is changing in time and this requires that the potential be time-dependent as well. The capacitance is a function of the geometry of the system and is constant unless the actual physical capacitor is altered, which it is not in this case. As figure 7.5 (a) shows, the potential across the resistor will be the same as that across the capacitor. Therefore, the charge on the capacitor is related to the current through the resistor by,

$$V(t) = I(t)R \quad \text{Eq. 7.4} \quad Q(t) = CV(t)$$

where the time dependencies are shown.

Current is defined as the time rate of change of electric charge, $I = dQ/dt$, so we can use (1) to obtain a differential equation for the charge.

$$Q = CR \frac{dQ}{dt} \quad (2)$$

We are trying to solve for the charge on the capacitor, however, so our $Q(t)$ represents the physical amount of charge on this capacitor. From the problem statement we know that this charge is decreasing (the statement says that the capacitor is **discharging**). Since the charge of the capacitor is decreasing we should write the current as, $I = -dQ/dt$. After making this adjustment it is possible to solve for the charge.

$$Q(t) = -CR \frac{dQ(t)}{dt} \quad (3)$$

$$\frac{dQ}{dt} = -\frac{1}{RC} Q \quad (4)$$
This is a differential equation to which the answer is known. You may just write it down, and I’ll write out every step as an example here,

\[ \frac{dQ}{dt} = -\frac{1}{RC} Q \]  
\[ \frac{dQ}{Q} = -\frac{dt}{RC} \]  
\[ \int \frac{dQ}{Q} = \int -\frac{dt}{RC} \]  
\[ \ln(Q) = -\frac{t}{RC} + \alpha \]  
\[ \exp(\ln(Q)) = \exp\left[-\frac{t}{RC} + \alpha\right] \]  
\[ Q = \exp\left[-\frac{t}{RC}\right] \exp(\alpha) \]  
\[ Q(t) = A \exp\left(-\frac{t}{RC}\right) \]

where in (8) the \( \alpha \) represents the constant of integration and the exponential of \( \alpha \) is written as \( A \) in (11).

It is acceptable for you to move straight to (11) because this differential equation solution is so common in physics that every one will believe you know it by heart.

\[ Q(t) = A \exp\left(-\frac{t}{RC}\right) \]

Solving for \( A \) requires using the initial condition on the system, which is that the initial charge on the capacitor is the total charge, \( Q(t = 0) = CV_o \).

\[ Q(t = 0) = CV_o = A \exp\left(-\frac{0}{RC}\right) \]  
\[ A = CV_o \]

\[ Q(t) = CV_o \exp\left(-\frac{t}{RC}\right) \]

and the charge decreases to zero as time goes to infinity.

The current has been defined as the negative time derivative of the charge, therefore it is given by,

\[ I(t) = -\frac{d}{dt} CV_o \exp\left(-\frac{t}{RC}\right) = -CV_o \frac{1}{RC} \exp\left(-\frac{t}{RC}\right) \]  
\[ I(t) = \frac{V_o}{R} \exp\left(-\frac{t}{RC}\right) \]

The current begins at its largest value and goes to zero as time progresses. This should make sound physical sense to anyone who has worked with capacitors in circuits. If you have a charged capacitor and then you short it out by placing a wire between the leads it produces a spark. If you quickly
remove this wire and then put it back you notice no additional spark. The capacitor partially dis-
charged to provide the first spark and now that there is considerably less charge in it the current it
can drive is also reduced.

(b) The problem requires that we use,

\[ W = \frac{1}{2} CV^2 \quad \text{Eq. 2.55} \]  

Calling the initial energy stored in the capacitor \( W_o \) and recalling that we are given the initial capac-
itor potential, \( V_o \),

\[ W_o = \frac{1}{2} CV_o^2 \quad \text{(19)} \]

To find the energy dissipated in the resistor we are asked to integrate the following,

\[ P = I^2 R \quad \text{Eq. 7.7} \]  

Power is the energy per unit time, so the energy is found through integration as suggested,

\[ \frac{dW}{dt} = P = I^2 R \rightarrow dW = I^2 R dt \]  

\[ W = \int_0^\infty I^2 Rd t = \int_0^\infty \frac{V_o^2}{R} \exp \left( -\frac{2t}{RC} \right) dt \]  

\[ = \frac{V_o^2}{R} \left( -\frac{RC}{2} \right) \left( \exp(-\infty) - \exp(0) \right) \]  

\[ = \frac{1}{2} CV_o^2 \quad \text{(24)} \]

which is equal to the energy originally stored in the capacitor, \( W_o \). Notice that the integration limits
are, \( 0 < t < \infty \). From part (a) we already know that the current will eventually reach zero, so you
could change these limits to go from, \( 0 < t < t_f \), where \( t_f \) is the time at which the current has
reached zero. This would require solving for \( t_f \), however, and that adds another step in the solution
to this problem. Since the current is zero after this time we can set the upper bound as infinity
without it contributing any energy to our expression.

(c) Connecting a battery to charge the capacitor results in a different description of the potential
in the circuit. The battery forces its potential across the resistor-capacitor combination, making the
total potential across those two elements equal to \( V_o \).

\[ V_{cap} + V_{res} = V_{batt} \quad \text{(25)} \]

\[ \frac{Q_{cap}}{C} + IR = V_o \quad \text{(26)} \]

where the potential across a resistor is given by, \( V = IR \) (Eq. 7.4).
Also in this problem, the charge on the capacitor is now increasing and the current should be written as, \( I = dQ/dt \). This presents the differential equation to solve.

\[
\frac{Q}{C} + R \frac{dQ}{dt} = V_o \tag{27}
\]

\[
\frac{dQ}{dt} = \frac{1}{R} \left( V_o - \frac{Q}{C} \right) \tag{28}
\]

\[
= \frac{1}{RC} (CV_o - Q) \tag{29}
\]

\[
\frac{dQ}{CV_o - Q} = \frac{dt}{RC} \tag{30}
\]

From this point the solution follows that of part (a) with the substitutions, \( \beta = CV_o - Q \), and \( d\beta = -dQ \). Put otherwise,

\[
- \ln(CV_o - Q) = \frac{t}{RC} + \text{constant} \tag{32}
\]

\[
CV_o - Q = A \exp \left( -\frac{t}{RC} \right) \tag{33}
\]

\[
Q = CV_o - A \exp \left( -\frac{t}{RC} \right) \tag{34}
\]

where \( A \) is again a constant to be determined.

Initial conditions require that \( Q(t = 0) = 0 \) because in this case the capacitor has no initial charge. Solving for \( A \),

\[
Q(t = 0) = 0 = CV_o - A \exp \left( -\frac{0}{RC} \right) \tag{35}
\]

\[
0 = CV_o - A \tag{36}
\]

\[
A = CV_o \tag{37}
\]

The charge is,

\[
Q(t) = CV_o \left[ 1 - \exp \left( -\frac{t}{RC} \right) \right] \tag{38}
\]

which reaches a final value of \( Q = CV_o \) as \( t \to \infty \).

The current is the time derivative of (38),

\[
I(t) = \frac{V_o}{R} \exp \left( -\frac{t}{RC} \right) \tag{39}
\]

Notice that the current still goes to zero as \( t \to \infty \). When the capacitor is fully charged there will be no more current in the system. Capacitors have a limit to the amount of charge that they can hold for a given potential, which is shown in (1).
(d) Use the given equation to find the energy output of the battery,

\[ W_{\text{batt}} = \int V_o I dt = \int_0^\infty V_o \cdot \frac{V_o}{R} \exp \left( -\frac{t}{RC} \right) dt \]

\[ = \frac{V_o^2}{R} (-RC) \left[ \exp(-\infty) - \exp(0) \right] \]

\[ = CV_o^2 \]

The energy dissipated in the resistor is found from (21), but the values of \( R \) and \( I^2 \) here are the same as in part (b) and therefore the energy is the same.

\[ W_{\text{res}} = \frac{1}{2} CV_o^2 \]

The final energy of the capacitor is given by,

\[ W = \frac{1}{2} CV^2 = \frac{1}{2} \frac{Q^2}{C} \]

The final value of \( Q \) is,

\[ Q(t \to \infty) = CV_o \left[ 1 - \exp(-\infty) \right] = CV_o \]

Inserting (45) into (44) gives the final energy of the capacitor,

\[ W_{\text{cap}} = \frac{1}{2C} \cdot C^2 V_o^2 = \frac{1}{2} CV_o^2 \]

Half of the work done by the battery (i.e. its expended energy) ends up stored in the capacitor.

[2.] Problem 7.7 in Griffiths

A metal bar slides on parallel conducting rails. The sliding is frictionless and the bars are separated by a distance \( l \). The rails are connected by a resistor of value \( R \), and there is a uniform background magnetic field \( \vec{B} \), which points into the page throughout the entire region. Reference figure 7.16 in Griffiths.

(a) The bar moves right at speed \( v \). Find the current in the resistor, and its direction.

(b) Find the magnetic force on the bar and its direction.

(c) The initial speed of the bar is \( v(t = 0) = v_o \). Find the speed at a later time, \( t \).

(d) Show that the total energy dissipated in the resistor is \( mv_o^2/2 \).

Solution

(a) The current in the resistor is given by, \( I = V/R \). Begin the solution by considering possible sources for this currents. In this problem the only potential is due to the emf, \( \mathcal{E} \) (the explanation for how \( V = \mathcal{E} \) may be found in Griffiths, page 293). To find the \( \mathcal{E} \) across a loop use,

\[ \mathcal{E} = -\frac{d\Phi}{dt} \quad \text{Eq. 7.13} \]

where \( \Phi \) is the magnetic flux through the surface enclosed by the loop.
In turn, $\Phi$ is found from,

$$\Phi = \int \vec{B} \cdot d\vec{a} \quad \text{Eq. 7.12}$$

where the integral is taken over the surface of interest.

Here the magnetic field is uniform, let $\vec{B} = B_o \hat{z}$, so the flux is this field multiplied by the area.

$$\Phi = B_o lx \quad \text{(49)}$$

where $x$ is defined as the horizontal length of the rectangular area between the sliding bar and the rest of the rest of the rail circuit. Since this flux is positive that means I have defined $d\vec{a}$ such that it points in the $\hat{z}$ direction parallel to $\vec{B}$ (the problem statement says that $\hat{z}$ is directed into the page). By the right-hand rule, defining the direction of $d\vec{a}$ in this way assumes that the current found is flowing in the clockwise direction. Placing your right thumb along the direction of $d\vec{a}$, your fingers curl in the direction of the supposed current. If the calculation of current comes out negative, then its direction will be opposite this supposed direction. We do not know the direction of the current until this value is determined.

Since $\hat{z}$ is into the page, an orthogonal system is obtained by defining $\hat{x}$ as pointing to the right (horizontal) and $\hat{y}$ pointing down (vertical).

Back to finding $E$,

$$E = -\frac{d\Phi}{dt} = -\frac{d}{dt}B_o lx \quad \text{(50)}$$

$$= -B_o l \frac{dx}{dt} \quad \text{(51)}$$

$$= -B_o l v \quad \text{(52)}$$

since $v = dx/dt$ by definition of $v$ being to the right, which is the same direction as $+\hat{x}$.

Also, now that $E$ has been determined it is possible to give the direction of the current. The current is proportional to $E$ ($I = E/R$), so a negative $E$ leads to a negative current. Our $E$ was defined by a clockwise loop; its negative value means the proper direction is counterclockwise. Following a counterclockwise loop in figure 7.16 means the current flows down the resistor.

The current is always represented by a positive value since we stated the direction, therefore,

$$I = \frac{B_o lv}{R} \quad \text{(53)}$$

(b) Recall that the force on a current in a magnetic field is,

$$\vec{F}_{mag} = \int I(d\vec{l} \times \vec{B}) \quad \text{Eq. 5.16}$$

where $d\vec{l}$ points in the direction of the current. The current is in the $-\hat{y}$ direction in the bar, according to the coordinate system I am using. In this problem all of the quantities are uniform, which allows,

$$\vec{F}_{mag} = I \vec{l} \times \vec{B} \quad \text{(55)}$$

$$= Il(-\hat{y}) \times B_o \hat{z} \quad \text{(56)}$$

$$= -IlB_o \hat{x} \quad \text{(57)}$$
Replace $I$ with (53), its value in terms of parameters given in the problem,

$$\mathbf{F}_{\text{mag}} = -\frac{l^2 B_0^2 v}{R} \hat{x}$$

(58)

this force is directed against the velocity of the bar.

(c) From (b) we solved for the magnetic force on the bar, which is acting to slow it down. We have,

$$\mathbf{F}_{\text{total}} = m\ddot{\mathbf{v}} = m \frac{d\mathbf{v}}{dt} = -\frac{l^2 B_0^2 v}{R} \hat{x}$$

(59)

The direction of $\mathbf{v}$ is given in the problem to be, by the coordinates used here, in the $\hat{x}$ direction. It is possible now to remove all vector dependence from the equation above.

$$\frac{dv}{dt} = -\frac{l^2 B_0^2 v}{mR}$$

(60)

Take the general solution of this differential equation (once again the same type of differential equation commonly found in physics, including problem [1.] of this assignment) and use the initial conditions to solve for the unknown constant, $A$,

$$v = A \exp \left( -\frac{l^2 B_0^2}{mR} t \right)$$

(61)

$$v(t = 0) = v_o = A \exp(0) \rightarrow A = v_o$$

(62)

$$v(t) = v_o \exp \left( -\frac{l^2 B_0^2}{mR} t \right)$$

(63)

So as we approach $t = \infty$ the velocity of the bar goes to zero. Once this velocity is zero there is no longer a change in the magnetic flux through the loop, meaning that there is no longer a force on the bar. Unless we push the bar again, it will come to a stop.

(d) Following the methods used previously to determine the energy delivered to a resistor, this energy here is,

$$W_{\text{res}} = \int I^2 R \, dt = \int_0^\infty \left( \frac{B_0 l^2 v^2}{R^2} \right) R \, dt$$

(64)

$$= \frac{B_0 l^2}{R^3} \int_0^\infty v_o^2 \exp \left( -\frac{2l^2 B_0^2}{mR} t \right)$$

(65)

$$= \frac{B_0 l^2 v_o^2}{R} \left( -\frac{mR}{2l^2 B_0^2} \right) \left[ \exp(-\infty) - \exp(0) \right]$$

(66)

$$= \frac{1}{2} m v_o^2$$

(67)

and the total energy delivered to the resistor is the expected value.

[3.] Problem 7.8 from Griffiths

A square loop of wire is a distance $s$ from an infinite wire of current $I$. The square loop has sides of length $a$. Reference figure 7.8 in Griffiths.
(a) Determine the magnetic flux through the loop.

(b) The loop is pulled directly away from the wire (entirely along the $s$ coordinate in the cylindrical system) at speed $v$. Find the emf generated in the wire and the direction of the current flow.

(c) What happens if the loop is moving to the right (along $z$ in the cylindrical coordinate system) at speed $v$ instead of along $s$ as in part (b)?

**Solution**

(a) The magnetic flux is found using (48). The magnetic field is that due to a very long (i.e. may be treated as infinite) wire carrying current $I$, \[
\vec{B} = \frac{\mu_0 I}{2\pi s} \hat{\phi} \quad \text{Eq. 5.41 (68)}
\]
where this is in cylindrical coordinates with the current directed along the $z$-axis.

Defining the current direction as above, the $+\hat{\phi}$ direction inside the square loop is coming out of the page. This is known because if you place your right thumb in the $+\hat{z}$ direction, then your fingers wrap around in the $+\hat{\phi}$ direction. This agrees with the field of $\vec{B}$ above.

The flux through the loop is, \[
\Phi = \int \vec{B} \cdot d\vec{a} = \int \frac{\mu_0 I}{2\pi s} \hat{\phi} \cdot ds \, dz \quad \text{(69)}
\]
\[
= \frac{\mu_0 I}{2\pi} \int_s^{s+a} \int_0^a \frac{1}{s} \, ds \, dz \quad \text{(70)}
\]
\[
= \frac{\mu_0 I a}{2\pi} (\ln(s+a) - \ln(s)) \quad \text{(71)}
\]
\[
= \frac{\mu_0 a I}{2\pi} \ln \left( \frac{s+a}{s} \right) \quad \text{(72)}
\]

While it is important to know the directions of the relevant vectors in determining the flux, this quantity is always a scalar value.

(b) Determining the emf is straightforward since we already have the flux. Note that our flux value in (72) defined the direction of $d\vec{a}$ to be $\hat{\phi}$. This is coming out of the page in the region of the loop. Using the right-hand rule with this direction gives a path going counterclockwise in the loop. If we get a positive emf in the next few steps, then the direction of the current it induces will also be counterclockwise in the loop. A negative emf means the current will be in the opposite direction.

The loop is moving away from the wire along $s$, therefore $s = s(t)$, \[
\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt} \frac{\mu_0 a I}{2\pi} \ln \left( \frac{s+a}{s} \right) \quad \text{(73)}
\]
\[
= -\frac{\mu_0 a I}{2\pi} \cdot \frac{d}{dt} \ln \left( \frac{s+a}{s} \right) \quad \text{(74)}
\]
\[
= -\frac{\mu_0 a I}{2\pi} \left[ \frac{s}{s+a} \cdot \left( \frac{1}{s} \frac{ds}{dt} + (s+a)(-s^{-2}) \frac{ds}{dt} \right) \right] \quad \text{(75)}
\]
\[
= -\frac{\mu_0 a I}{2\pi} \left[ \frac{1}{s+a} \frac{ds}{dt} \left( 1 - \frac{s+a}{s} \right) \right] \quad \text{(76)}
\]
\[ = -\frac{\mu_0 a I}{2\pi} \left[ \frac{1}{s + a} \frac{ds}{dt} \left( -\frac{a}{s} \right) \right] \quad (77) \]
\[ = \frac{\mu_0 a^2 I v}{2\pi s (s + a)} \quad (78) \]

since \( ds/dt = v \) by definition. Notice that in this case the emf depends not only on the velocity, but also on the position. Once the loop is very far away from the current-carrying wire \( (s \rightarrow \infty) \), the magnetic field is essentially zero and there is no magnetic flux through the loop. Regardless of the velocity at this location, the magnetic flux does not change from its zero value. The emf should be expected to reach zero once the distance away from the current gets very large.

This emf is positive, therefore our direction choices for the flux calculation tells us that the current induced in the loop will be counterclockwise.

c) The velocity of the loop in this part is directed along the \( z \) axis. The \( s \) position is therefore constant, \( s \neq s(t) \). The flux through the loop is unchanged, but now the time derivative of this flux is zero. There is no emf in this case.

[4.] Problem 7.11 from Griffiths

A square loop of aluminum is placed with its top half in a region of uniform magnetic field, \( \vec{B} \). This field points into the page. The loop is released and falls under the influence of gravity. Reference figure 7.19 in Griffiths. For the following tasks use a field of \( 1T \), mass density of aluminum is \( \rho = 2.7 \times 10^3 \text{ kg/m}^3 \), and the resistivity of aluminum is (according to table 7.1 in Griffiths) \( \eta = 2.65 \times 10^{-8} \Omega \text{m} \).

- Calculate the terminal velocity of the loop. Compute a numerical value in units of \( m/s \).
- Determine the velocity of the loop, \( v = v(t) \).
- Calculate the time taken for the loop to reach 90% of its terminal velocity. Compute a numerical value in units of \( s \).
- State what happens if the loop is cut such that the circuit is broken.

Solution

Let the length of the sides of the loop be \( l \). Solving for the motion of the loop requires solving,

\[ \vec{F}_{\text{tot}} = m\vec{a} = m\vec{g} + \vec{F}_{\text{mag}} \quad (79) \]

where \( \vec{g} \) is the acceleration due to gravity and \( \vec{F}_{\text{mag}} \) is the magnetic force experienced by the loop.

The coordinate system used here is \(+\hat{z}\) into the page, so that \( \vec{B} = B_0 \hat{z} \), the \(+\hat{x}\) direction is to the right, and the \(+\hat{y}\) direction is downward.

With the uniform field and the discrete length through which current may flow, the magnetic force is given by (55). To find the current we must first know the emf. The emf is found similarly to the method used in problem 7.7.

The magnetic flux through the loop is,

\[ \Phi = B_0 ly \quad (80) \]
The emf is,

\[ \mathcal{E} = -\frac{d}{dt}B_0ly = B_0lv \quad (81) \]

where \( dy/dt = -v \) since the velocity is along the \( +\hat{y} \) but the \( y \) length is decreasing as the loop falls.

The emf is positive and the loop was determined using \( d\vec{a} \) in the \( \hat{\phi} \) direction. This implies a current going clockwise in the loop, which is in the \( +\hat{x} \) direction in the top length of the loop. The magnitude of this current is,

\[ I = \frac{\mathcal{E}}{R} = \frac{B_0lv}{R} \quad (82) \]

where \( R \) is the resistance of the loop.

Now we are ready to use (55).

\[ \vec{F}_{mag} = I(\vec{l} \times \vec{B}) = \frac{B_0lv}{R}(\hat{l} \times \vec{B}_0\hat{z}) \quad (83) \]

\[ = -\frac{B_0^2l^2v}{R}\hat{y} \quad (84) \]

According to the coordinates used here, this magnetic force is acting against the gravitational force.

The terminal velocity of the loop occurs when the acceleration is zero because the gravitational and magnetic forces balance out. Setting \( \vec{a} = 0 \) lets us change \( v \rightarrow v_t \) in the force equation, (79), where \( v_t \) is the terminal velocity. Back to (79), using \( \vec{g} = g\hat{y} \),

\[ 0 = m\vec{g} + \vec{F}_{mag} \quad (85) \]

\[ -mg = -\frac{B_0^2l^2v}{R}\hat{y} \quad (86) \]

\[ mg = \frac{B_0^2l^2v}{R}v_t \quad (87) \]

\[ v_t = \frac{gmR}{B_0^2l^2} \quad (88) \]

To get an actual number for this velocity we need to define the resistance and mass of the loop. The resistance of an object with uniform cross section is,

\[ R = \frac{\eta L}{A} \quad (89) \]

where \( L \) is the length of the object and \( A \) is the area of the cross section.

In this problem we have, noting that there are four segments of the loop,

\[ R = \frac{4l\eta}{A} \quad (90) \]

where the area is left unknown. Griffiths hints in the problem statement that the dimensions will cancel out, so there is good reason to believe this \( A \) will not matter for the final answer.
The mass of the loop is given by its volume multiplied by the mass density. Again, the following factor of four relates to the four lengths that comprise the loop,

\[ m = 4lA\rho \quad (91) \]

where this \( A \) is the same cross sectional area found in the resistance calculation.

Return to the expression for terminal velocity and use (90) and (91),

\[
v_t = \frac{g m R}{B_0^2 l^2} = \frac{(9.8 \text{ m/s})(4lA\rho)(\frac{4\eta}{A})}{1 \text{ T}^2 \text{ m}^2} \quad (92)
\]

\[
= (9.8)(16)\rho\eta \frac{\text{kg} \Omega}{\text{T}^2 \text{s}^2 \text{m}} \quad (93)
\]

\[
= (9.8)(16)(2.7 \times 10^3)(2.65 \times 10^{-8}) \text{ m/s} \quad (94)
\]

\[
= 1.12 \times 10^{-2} \text{ m/s} \quad (95)
\]

The terminal velocity of the loop is approximately 1.12 cm/s.

It is not obvious that the units in (93) simplify to m/s, so here is a proof. It begins by writing the resistance and magnetic field units in terms of the fundamental units (i.e. meters, seconds, Coulombs, kilograms).

\[
\frac{\text{kg} \Omega}{\text{T}^2 \text{s}^2 \text{m}} = \frac{\text{m}}{\text{s}} \quad (96)
\]

\[
\Omega = \frac{V}{A} \quad (97)
\]

\[
V = \frac{J}{C} = \frac{\text{N m}}{\text{C}} = \frac{\text{kg m}}{\text{s}^2 \text{m}} = \frac{\text{kg m}^2}{\text{C} \text{s}^2} \quad (98)
\]

\[
A = \frac{C}{s} \quad (99)
\]

\[
\therefore \quad \Omega = \frac{\text{kg m}^2}{\text{C}^2 \text{s}} \quad (100)
\]

The magnetic field unit, the Tesla, becomes,

\[
T = \frac{\text{N}}{\text{A m}} = \frac{\text{kg m}}{\text{s}^2 \text{C} \text{m}} = \frac{\text{kg}}{\text{C} \text{s}} \quad (101)
\]

Bringing all of this together gives,

\[
\frac{\text{kg} \Omega}{\text{T}^2 \text{s}^2 \text{m}} = \frac{\text{kg} \frac{\text{kg m}^2}{\text{C}^2 \text{s}}}{\text{kg} \frac{\text{kg m}^2}{\text{C}^2 \text{s}^2 \text{m}}} = \frac{\text{m}}{\text{s}} \quad (102)
\]
Solving for the velocity requires returning to (79) and using $\ddot{a} \neq 0$. The mathematics follows the method used in problem 7.2, part (c).

\begin{align}
\frac{dv}{dt} &= g - \frac{B_o^2 l^2}{mR} v \\
\frac{dv}{g - \frac{B_o^2 l^2}{mR} v} &= dt \\
-\frac{mR}{B_o^2 l^2} \ln \left( g - \frac{B_o^2 l^2}{mR} v \right) &= t + \text{constant} \\
\ln \left( g - \frac{B_o^2 l^2}{mR} v \right) &= -\frac{B_o^2 l^2}{mR} t + \text{constant} \\
g - \frac{B_o^2 l^2}{mR} v &= A \exp \left( -\frac{B_o^2 l^2}{mR} t \right) \\
v(t) &= \frac{gmR}{B_o^2 l^2} - A \exp \left( -\frac{B_o^2 l^2}{mR} t \right)
\end{align}

where $A$ is a constant.

Use the initial conditions to find $A$. The loop is released after being placed partially in the magnetic field. We may say, $v(t = 0) = 0$. Then,

\begin{align}
v(t = 0) &= 0 = \frac{gmR}{B_o^2 l^2} - A \exp(0) \\
A &= \frac{gmR}{B_o^2 l^2}
\end{align}

Replacing $A$ gives the velocity as,

\begin{align}
v(t) &= \frac{gmR}{B_o^2 l^2} \left[ 1 - \exp \left( -\frac{B_o^2 l^2}{mR} t \right) \right]
\end{align}

Finding the time at which 90% of the terminal velocity is reached is easy if we notice that the expression for the velocity contains the terminal velocity as a factor.

\begin{align}
v(t) &= \frac{gmR}{B_o^2 l^2} \left[ 1 - \exp \left( -\frac{B_o^2 l^2}{mR} t \right) \right] \\
&= v_t \left[ 1 - \exp \left( -\frac{B_o^2 l^2}{mR} t \right) \right]
\end{align}

The velocity has reached 90% of the terminal velocity when,

\begin{align}
\frac{v}{v_t} = 0.9
\end{align}
This time occurs at,

\[ \frac{v}{v_t} = 0.9 = 1 - \exp\left( -\frac{B_o^2 t^2}{mR} \right) \]  

(116)

\[ \exp\left( -\frac{B_o^2 t^2}{mR} \right) = 0.1 \]  

(117)

\[ -\frac{B_o^2 t^2}{mR} = \ln(0.1) \]  

(118)

\[ t = -\frac{mR \ln(0.1)}{B_o^2 t^2} \]  

(119)

Insert (90) and (91) into (119) to get a numerical value for the time,

\[ t = -\frac{(4Al\rho)(4\ln{\frac{4\eta}{\lambda}}) \ln(0.1)}{I^2} \]  

(120)

\[ = -16(2.7 \times 10^3)(2.65 \times 10^{-8})(-2.303) \]  

(121)

\[ = 2.64 \times 10^{-3} \text{ s} \]  

(122)

where the units are simply given and not proven.

Finally, cutting a slit in the ring breaks the circuit and prevents current from flowing. Without current there can be no magnetic force on the loop. In this case the loop falls according to gravity, with no effect from the magnetic field (aluminum is a non-magnetic metal).

[5.] Problem 7.12 from Griffiths

A long solenoid is carrying an alternating current. The radius of the solenoid is, \( a \), and the field inside due to its current is \( \vec{B}(t) = B_o \cos(\omega t) \hat{z} \). A loop of wire is placed coaxially inside the solenoid. The radius of this loop is \( a/2 \) and its resistance is \( R \). Determine the current in the loop, \( I = I(t) \).

**Solution**

The current will be found through the usual \( I = \mathcal{E}/R \) relation. Since the loop is coaxial with the solenoid it is a good idea to let the \( d\vec{a} \) of the loop point in the \( \hat{z} \) direction so as to be parallel with the magnetic field.

According to (47), the emf is,

\[ \mathcal{E} = \frac{d}{dt} \int_0^{a/2} \int_0^{2\pi} B_o \cos(\omega t) \hat{z} \cdot s \, ds \, d\phi \hat{z} \]  

(123)

\[ = -\frac{d}{dt} B_o \cos(\omega t) (2\pi) \left( \frac{s^2}{2} \right)_{0}^{a/2} \]  

(124)

\[ = -\frac{\pi a^2 B_o}{4} (-\omega \sin(\omega t)) \]  

(125)

\[ = \frac{\pi a^2 \omega B_o}{4} \sin(\omega t) \]  

(126)
The current is,

\[ I(t) = \frac{\pi a^2 \omega B_0}{4R} \sin(\omega t) \]  

(127)

[6.] **Problem 7.16 from Griffiths**

A current \( I = I_o \cos(\omega t) \) flows in a wire. This current returns along a concentric tube outside the wire.

(a) What is the direction of the induced electric field (radial, longitudinal, or circumferential)?

(b) Assume \( \vec{E}(s \to \infty) = 0 \). Find the electric field, \( \vec{E}(s, t) \).

**Solution**

(a) Note that Griffiths is sure to point out the similarity between Ampere’s law and Faraday’s law (page 306). The electric field in this problem can be found from,

\[ \oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt} \]  

Eq. 7.18

(128)

Ampere’s law is,

\[ \oint \vec{B} \cdot d\vec{l} = \mu_o I_{enc} \]  

(129)

In the wire there is only magnetic field between the conductors. We can use Ampere’s law as in previous assignments to get the field inside the coaxial cable as,

\[ \vec{B} = \frac{\mu_o I_o}{2\pi s} \hat{\phi} = \frac{\mu_o I_o \cos(\omega t)}{2\pi s} \hat{\phi} \]  

(130)

This means that the magnetic flux is determined through surfaces that have vector directions parallel to \( \hat{\phi} \), which is itself parallel to \( \vec{B} \) (i.e. those surfaces with \( d\vec{A} \parallel +\hat{\phi} \)). From Ampere’s law, if an enclosed current was along the \( \hat{\phi} \) direction, then the right-hand rule would tell us that the magnetic field it induces is along the \( z \)-axis. Here the changing flux is “along” the \( +\hat{\phi} \) direction and therefore the electric field it induces is in the \( z \) direction.

*The electric field is in the \( \hat{z} \) direction, longitudinal.*

Note that the magnetic flux is not a vector; it has no direction. There is, however, a choice in direction made when computing the flux through a surface. It is this direction that you should keep in mind when thinking about the direction of the induced electric field.

(b) To get the electric field we first determine the magnetic flux.

\[ \Phi = \oint \vec{B} \cdot d\vec{A} = \oint \frac{\mu_o I_o \cos(\omega t)}{2\pi s} \hat{\phi} \cdot ds \, dz \, \hat{\phi} \]  

(131)

\[ = \frac{\mu_o I_o \cos(\omega t)}{2\pi} \int_s^a \int_0^l ds \, dz \]  

(132)

\[ = \frac{\mu_o I_o \cos(\omega t)}{2\pi} l \ln \left( \frac{a}{s} \right) \]  

(133)

Notice that the area in which the magnetic flux is changing is given by \( A = l(a - s) \), which explains the limits of the integration in (132).
Now the electric field can be found using (128). The electric field is already known to be in the $\hat{z}$ direction, which accounts for the simplification in the left side of the following step,

$$E_z l = -\frac{d}{dt} \frac{\mu_0 I_0 \cos(\omega t)}{2\pi} l \ln \left( \frac{a}{s} \right)$$  \hspace{1cm} (134)

$$E_z = -\frac{\mu_0 I_0}{2\pi} \ln \left( \frac{a}{s} \right) (-\omega \sin(\omega t))$$  \hspace{1cm} (135)

$$= \frac{\mu_0 I_0 \omega \sin(\omega t)}{2\pi} \ln \left( \frac{a}{s} \right)$$  \hspace{1cm} (136)

The geometry used in applying Faraday’s law to this problem is given in figure 1. This is similar to using Ampere’s law while solving for the magnetic field of a solenoid. The magnetic field due to the current in the center wire is shown. This field changes direction in time. The slightly shaded region is the area through which the magnetic flux changes. As the problem statement says, the electric field may be assumed to be zero as $s \to \infty$. This means we may neglect the contribution of $\vec{E}$ along the vertical length of the loop at the far right.

The electric field is,

$$\vec{E} = \frac{\mu_0 I_0 \omega \sin(\omega t)}{2\pi} \ln \left( \frac{a}{s} \right) \hat{z}$$  \hspace{1cm} (137)

Figure 1: Geometry for using Faraday’s law to find the electric field in problem 7.16.
Problem 7.17 from Griffiths

A long solenoid has radius $a$ and $n$ turns per unit length. This solenoid passes through a loop of wire with resistance $R$. Reference figure 7.27 in Griffiths.

(a) Represent the current in the solenoid, $I$, by $dI/dt = k$ where $k$ is a constant. Find the current that flows in the loop. Does it flow to the right or left through the resistor in figure 7.27?

(b) Now let the solenoid current remain constant. You pull the solenoid out from the loop, turn it around, and then reinsert it through the loop. What total charge passes through the resistor in this process?

Solution

(a) To find the current induced in the loop we will have to find the emf generated by the solenoid. There is no magnetic field outside the solenoid. The field inside the solenoid is given by,

$$
\vec{B} = \mu_0 n I \hat{z}
$$

Eq. 5.57 (138)

where $I$ arbitrarily assign the direction.

This field is uniform. The magnetic flux is the flux through the surface area of the solenoid.

$$
\Phi = B \cdot \text{area} = \mu_0 n I \pi a^2
$$

(139)

The emf is,

$$
\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt} \mu_0 n I \pi a^2
$$

(140)

$$
= -\mu_0 n \pi a^2 \frac{dI}{dt}
$$

(141)

$$
= -\mu_0 n \pi a^2 k
$$

(142)

The definition of $\vec{B}$ and the area vector being in the $+\hat{z}$ direction meant that the direction of the current derived from the result above should be in the $+\hat{\phi}$ direction. Note from figure 7.27 that the current in the solenoid is flowing in the $+\hat{\phi}$ direction. The negative emf found above requires that the current induced in the loop must be opposite the solenoid current. The induced current flows to the right across the resistor.

One could also use Lenz’s law to determine the direction of the current in the loop. The current in the solenoid is increasing over time, therefore the magnetic field in the solenoid is as well. The surface areas of the solenoid and wire loop are constant so the magnetic flux through the loop with the resistor must be increasing. The current induced in the loop will try to counteract this flux change by generating a magnetic field against the change. The induced magnetic field tries to cancel out the increasing field by directing itself oppositely. To get a magnetic field in the $-\hat{z}$ direction, the current in the loop must flow to the right through the resistor.

The value of the current is,

$$
I = \frac{\mathcal{E}}{R} = \frac{\mu_0 n \pi a^2 k}{R}
$$

(143)

(b) The change in magnetic flux through the loop does not change uniformly in time. Fortunately, it does not matter how quickly the flux changes, the total change in flux results in a specific amount
of charge flow regardless of the time over which it changes. This is somewhat discussed in example 7.8 of Griffiths.

Solve this problem by treating all of the parameters as discrete. This means that we are not concerned with how things change through time, rather, we are concerned with the total changes of the relevant quantities.

In a discrete consideration we have a current,

\[ I = \frac{\Delta Q}{\Delta t} \]  \hspace{1cm} (144)

This \( \Delta Q \) is the total amount of charge that passes through the resistor in a time \( \Delta t \).

Continuing this discrete analysis,

\[ \Delta Q = I \Delta t \]  \hspace{1cm} (145)

\[ = \frac{E}{R} \Delta t \]  \hspace{1cm} (146)

\[ = -\frac{1}{R} \frac{\Delta \Phi}{\Delta t} \Delta t = -\frac{\Delta \Phi}{R} \]  \hspace{1cm} (147)

where (147) shows that the time over which this change occurs (\( \Delta t \)) does not affect the change in the total charge.

We need to find the total change in the magnetic flux due to the removing and replacing of the solenoid.

\[ \Delta \Phi = \Phi_{\text{final}} - \Phi_{\text{initial}} \]  \hspace{1cm} (148)

The current (and hence the magnetic field) in the solenoid remains constant. The initial and final fluxes will be equal in magnitude but have different signs due to the orientation of the solenoid. We have,

\[ \Phi_{\text{initial}} = \mu_0 n I \pi a^2 \]  \hspace{1cm} \Phi_{\text{final}} = -\mu_0 n I \pi a^2 \]  \hspace{1cm} (149)

The total change in magnetic flux is,

\[ \Delta \Phi = -2\mu_0 n I \pi a^2 \]  \hspace{1cm} (150)

The total charge passing through the resistor is,

\[ \Delta Q = -\frac{\Delta \Phi}{R} \]  \hspace{1cm} (151)

\[ = \frac{2\mu_0 n I \pi a^2}{R} \]  \hspace{1cm} (152)

[8.] **Professor Carter Problem**

A metal disk of radius \( a \), thickness \( d \), and conductivity \( \sigma \) is placed at the center of a long solenoid of radius \( b > a \) (i.e. the axis of the disk is also the axis of the solenoid). The current in the solenoid is time dependent: \( I(t) = I_o \sin(\omega t) \). Find the induced current density as a function of space and
time in the disk. You may assume the magnetic field is quasi-static (i.e. you can use magnetostatic solutions for the magnetic field of the solenoid).

**Solution**

We want to determine the current density in this conducting disk. From a conceptual standpoint, since it takes electric fields to drive currents we should expect that the best way to solve this problem is to find the induced electric field in the disk and then relate that to a current density. Such a relation has been provided by Griffiths,

\[ \vec{J} = \sigma \vec{E} \quad \text{Eq. 7.3} \]  

The electric field in the disk is generated by the changing magnetic flux of the solenoid. Solving for this electric field follows the same general method applied elsewhere in this assignment. We are allowed to use magnetostatic solutions for the magnetic field due to the current in the solenoid. As referenced in (138) we have for the field inside the solenoid,

\[ \vec{B}(t) = \mu_0 n I(t) \hat{z} \quad \text{(154)} \]

where \( n \) is the turns per unit length of the solenoid (this is not explicitly given in the problem, so any variable you use to describe the turn density of the solenoid is acceptable).

Use (128) to solve for the electric field inside the disk. This is the field induced by the changing magnetic flux of the solenoid through this disk. Figure 2 illustrates the geometry used in the steps below. The magnetic flux is directed along \( \hat{z} \), so I choose an Amperian loop that is directed along \( +\hat{\phi} \).

\[ E_{\phi}(2\pi s) = -\frac{d\Phi}{dt} \quad \text{(156)} \]

\[ E_{\phi} = \frac{1}{2\pi s} \cdot -\frac{d}{dt} [\mu_0 n I_o \sin(\omega t) \pi s^2] \quad \text{(157)} \]

where (157) takes advantage of the fact that the magnetic field is uniform in space and the magnetic flux is therefore given by the magnetic field times the area.

\[ E_{\phi} = -\frac{\mu_0 n I_o \omega s}{2} \cos(\omega t) \quad \text{(158)} \]

The electric field in the disk is,

\[ \vec{E} = -\frac{\mu_0 n I_o \omega s}{2} \cos(\omega t) \hat{\phi} \quad \text{(159)} \]

Referring to (153), the current density in the disk is,

\[ \vec{J} = -\frac{\mu_0 n I_o \omega s}{2} \cos(\omega t) \hat{\phi} \quad \text{(160)} \]

Notice that the current in the disk is directed along \( -\hat{\phi} \). Using the right-hand rule, the magnetic field generated by this current will be directed oppositely the changing magnetic field of the solenoid. This is a good reason to have confidence in our solution.
Figure 2: Top view of the disk inside the solenoid for problem [8.]. The dot in the center represents the $+\hat{z}$ direction coming out of the page. The conducting disk is shaded grey and has a radius $a$. The solenoid’s radius is $b$. The dashed line is the “Amperian” loop used to determine the induced electric field. Based on the $z$ axis and direction, $+\phi$ is directed as shown by the arrows on the dashed line.

This problem was solved without worrying about the specific resistance of the disk. The final answer, however, does include this physical dependence because of the appearance of the $\sigma$ term.

In light of the lectures that have been given after this homework was assigned, we know that this solution is not strictly correct. The current density of the disk generates a magnetic field that contributes to the total flux through the disk. Since the current density in (160) is time dependent the magnetic flux it contributes will lead to an emf in the disk. We solved this problem while ignoring the self-inductance of the disk.